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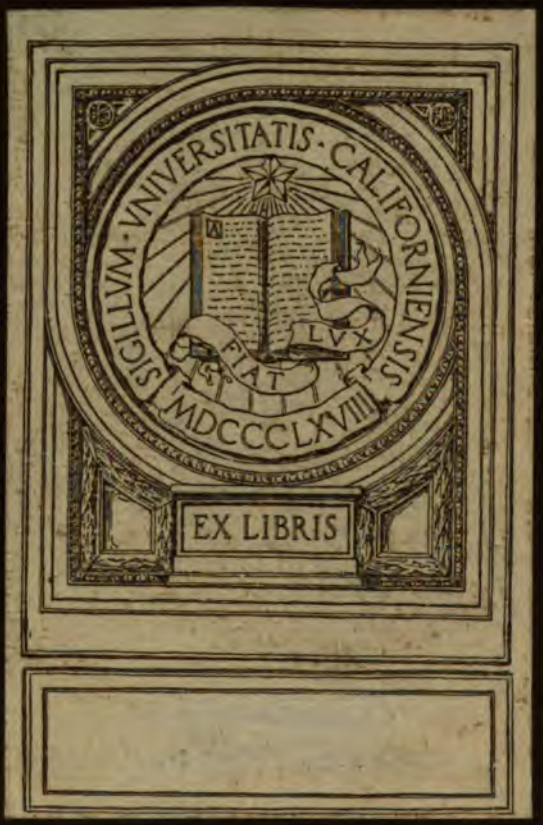
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SUSPENSION BRIDGES

ARCH RIBS AND

CANTILEVERS

BY

WM. H. BURR, C.E.

Professor of Civil Engineering in Columbia University, and Consulting Engineer; Member of the American Society of Civil Engineers; Member of the Institution of Civil Engineers London; Honorary Member of the Society of Civil Engineers of the Republic of Panama; etc., etc.

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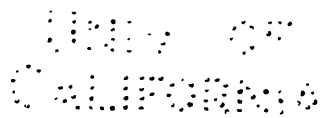

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PREFACE

THIS book has been written primarily to meet the author's needs in the class-room, where the chief requisite is the clear elucidation of general principles in connection with their bearing upon engineering work. The satisfactory accomplishment of such a task is none too easy in the treatment of the more simple statically determinate trusses or other structures, but it is excessively difficult when some statically indeterminate structures are under consideration. The endeavor has been made to give each main structure a general treatment so as to make one demonstration cover all desired or useful special cases, as conducing to both brevity and clearness. Unfortunately, such composite structures as the stiffened suspension bridge cannot be treated with the thoroughness required by modern engineering practice without formidable analysis, but great pains have been taken by the author to put formulæ in such shape as to make them applicable with the least possible labor, particularly by the aid of the many numerical tables given in the text. In consequence of this latter feature, it is hoped that the book may be found useful to professional practitioners.

There are shown in Chapters IV. and V. the two prominent methods of analysis employed at the present time for the treatment of stiffened suspension bridges. The method based upon the principle of Least Work involves the simplest equations, as all exponential quantities are avoided, but the initial equations on which is based the general value of the horizontal component of cable stress do not involve specifically the deflection of the cables produced by the stresses to be determined, while the method of deflections does explicitly provide for that deflection in its initial equations. For this reason the method of Least Work has been considered by some less accurate than the method by deflections.

As a matter of abstract analysis the justness of this view may be recognized, but it does not necessarily follow that the

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results obtained by the two different methods must greatly differ. It is a comparatively simple matter to determine by a sufficiently close approximation the deflection of the cable under a given system of loading. If that deflection, positive or negative, be algebraically added to the centre sag of the cable, thus affording a value of the centre sag under the loading to be treated, the Method of Least Work may give results equally accurate with those of the method of deflections, or possibly even more accurate.

The assumption in the method of deflections that all moving load is uniformly distributed over the span in determining the expression for the work involved through vertical deflection, as a general procedure, is certain open to serious criticism, as will be indicated. A full consideration, therefore, of all the features of the two methods justifies the use of the Method of Least Work, especially with the adjusted centre sag, as being at least equally accurate with the method of deflections, while its equations are certainly simpler and more easily workable.

It will be obvious even in a casual examination of the book that Chapter III. and the first part of Chapter IV. follow closely Professor Melan's excellent work. The author further takes pleasure in acknowledging his obligations to Mr. L. Moiseieff, C.E., and Mr. M. Zaslaw, C.E., of the Department of Bridges of New York City, the latter of whom has been of much assistance in the matter prepared in connection with the method of treatment of arch ribs based on the principle of Least Work. Messrs. J. K. Finch, C.E., and L. D. Northsworthy, C.E., have also rendered service, in the preparation of plates and in the reading of manuscript, of substantial value and great relief to the author.

Prof. B. O. Peirce, of Harvard University, has also kindly aided the author at points where his mathematical skill and experience have been of much service.

The aid of these professional friends has made the preparation of the manuscript possible in the midst of professional engagements of such character as to have rendered the accomplishment of the work otherwise impossible.

WM. H. BURR.

Columbia University,
August 1, 1913.

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SUSPENSION BRIDGES

CHAPTER I*

Art. 1.—Approximate Theory of the Stiffened Suspension Bridge with Parabolic Cable.

THE analyses of this chapter are based wholly upon the assumption that the problems treated are statically determinate. They are sufficiently close to serve the purposes of approximate computations and preliminary estimates. They are useful for a wide range of work, but for final design and other accurate operations the more exact analyses of subsequent chapters of this book should be employed. In the latter analyses it is sometimes required to solve equations by trial, and in such analytic operations approximate values of the quantities sought may easily be found by the formulæ of this chapter. Inasmuch as this class of analytic operations is somewhat extended, and as it includes a large number of formulæ which it is desirable to place in as convenient form as possible, this chapter will be made complete in all its demonstrations, independently of the much more general demonstrations which follow.

Two methods of loading will be taken in each case, *viz.*, a uniform load passing on the bridge from one end of the span, and a single weight or load placed at any point. The last method alone is equivalent to a general treatment for any loading whatever.

---

*The notation in this Chapter is not uniform with that of the subsequent Chapters.

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Art. 2.—Curve of Cable for Uniform Load per Unit of Span—Suspension Rods Vertical—Heights of Towers, Equal or Unequal—Generalization.

In the figure, let $E\,H'\,C$ represent the cable of a suspension bridge carrying a load extending over the whole span. In the ordinary experience of an engineer, the load carried by a suspension-bridge cable, including the stiffening-truss, is nearly uniform in intensity in reference to a horizontal line; so nearly uniform per foot of span, in fact, that it is assumed to be exactly so, and such an assumption will be made in the present instance.

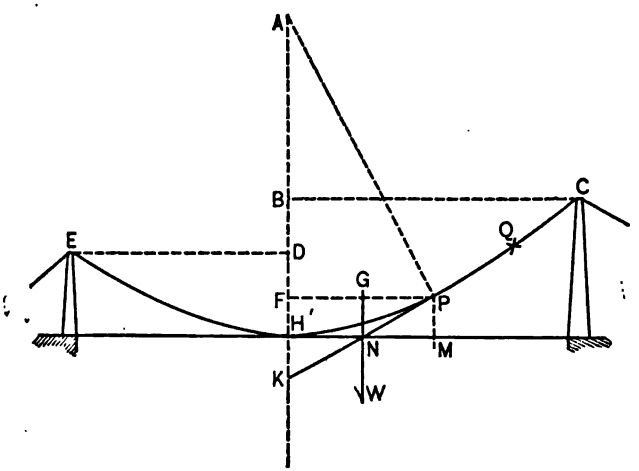

FIG. 1.

Let $(E\,D + B\,C) = l$ = span; $B\,H' = f_1$; $D\,H' = f_2$; w = load per horizontal foot, and let x be measured horizontally from H', the lowest point of the cable. The height of the highest tower above the lowest point of the cable is f_1, and that of the other f_2.

The ordinate of any point P is x, the load on $H'\,M$ is, consequently, $W = w\,x$. Draw $P\,K$ tangent to the curve at P, then, by the first principles of statics, it is known that the direction of the cable tensions at P and H' and the direction

of W must intersect in one point N. Since, however, w is uniform along x, the resultant direction of W passes through N, half way between H' and M. Hence $F H' = H' K$; or, since $F K$ is the subtangent, *the abscissa, $F H'$, of the curve is equal to half the subtangent*, consequently the curve is the ordinary parabola.

Again, the horizontal component of the tension of a cable must be a constant quantity if the loading (as in the present case) be wholly vertical; let H be that component.

Let $G N P$ be taken for the triangle (right-angled) of forces at P, in which $N P$ represents the cable tension at P, $G N$ the load, $W = w x$, and $G P$ the constant horizontal component H.

Then let $A P$ be drawn normal to the curve at P; the triangles $A F P$ and $G N P$ will be similar. There can now be at once written the relation:

$$\frac{A F}{G P} = \frac{F P}{G N} = \frac{x}{wx} = \frac{1}{w};$$

but

$$G P = H \therefore A F = \frac{H}{w} = \text{constant} \quad . \quad . \quad . \quad 1.$$

$A F$ is the subnormal of the curve of the cable, and, since it is constant, the curve is the ordinary parabola.

The preceding results may be generalized in a very simple and easy manner.

If any two points, as P and Q, be considered fixed, and if the portion $P Q$ of the cable carry the same intensity of load w as before, there will at once result the general case of a flexible cable carrying a load whose intensity, along a straight line, and direction are uniform. There may then be stated the general principle: *If a perfectly flexible cable carry a load uniform in direction and intensity in reference to a straight line, the cable will assume the form of an ordinary parabola whose axis will be parallel to the direction of the loading.*

This principle finds its application in the case of a suspension bridge with inclined, but parallel, suspension rods.

Art. 3.—Parameter of Curve—Distance of Lowest Point of Cable from either Extremity of Span—Inclination of Cable at any Point.

Attending to the figure and notation of the previous Article, the equation of the curve, the origin of co-ordinates being taken at H', is:

$$x^2 = 2\,p\,y;$$

in which $2\,p$ is the parameter.

Let $B\,C = x_1$ and $E\,D = x_2$, then there may be written $x_1^2 = 2\,p\,f_1$, $x_2^2 = 2\,p\,f_2$, and $2\,x_1\,x_2 = 4\,p\sqrt{f_1 f_2}$.

Hence:

$$(x_1 + x_2)^2 = l^2 = 2p\,(\sqrt{f_1} + \sqrt{f_2})^2 = 2p\,(f_1 + 2\sqrt{f_1 f_2} + f_2) \quad . \quad 2.$$

$$\therefore p = \frac{l^2}{2\,(\sqrt{f_1} + \sqrt{f_2})^2} = \frac{l^2}{2\,(f_1 + 2\sqrt{f_1 f_2} + f_2)} \quad . \quad 3.$$

If the towers are of the same height, $f_1 = f_2 = f$ and

$$p = \frac{l^2}{8f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 4.$$

It will be shown in the next Article that the horizontal component H of the cable stress is $H = w\,p$. Hence:

$$x^2 = 2\,p\,y = \frac{2\,H}{w}\,y \quad . \quad . \quad . \quad . \quad . \quad 4a.$$

As x_1 is the horizontal distance from the lowest point of the cable to that end of the span at which f_1 is found, *i. e.*, $B\,C$ in the figure, while x_2 is the other segment of the span, by the equations immediately preceding:

$$x_1 = \frac{l\sqrt{f_1}}{\sqrt{f_1} + \sqrt{f_2}} \quad . \quad . \quad . \quad . \quad . \quad 5.$$

$$x_2 = \frac{l\sqrt{f_2}}{\sqrt{f_1} + \sqrt{f_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad 6.$$

$$\text{If } f_1 = f_2;\ x_1 = x_2 = \frac{l}{2} \quad . \quad . \quad . \quad . \quad 7.$$

Referring to the figure, since $K\,H' = H'\,F = y$, if i is the

inclination to a horizontal line, of the curve at any point P, then $x \tan i = 2y$;

$$\text{hence, } \tan i = \frac{2y}{x} \therefore \sec i = \sqrt{1 + \frac{4y^2}{x^2}}. \quad . \quad . \quad 8.$$

At the tops of the towers:

$$\tan i_1 = \frac{2f_1}{x_1} \text{ and } \tan i_2 = \frac{2f_2}{x_2} \quad . \quad . \quad . \quad . \quad 9.$$

$$\text{If } f_1 = f_2, \tan i_1 = \tan i_2 = \frac{4f}{l} \quad . \quad . \quad . \quad . \quad 10.$$

Art. 4.—Resultant Tension at any Point of the Cable.

In the first Article of this Chapter there was recognized the general principle that if the loading on a cable is uniform in direction, the component of cable tension normal to that direction will be constant at all points of the cable. In the present case the resultant tension at the lowest point of the cable will be this constant component H.

From Eq. 1, $H = w A F$. $A F$ is the subnormal of the curve and it is known to be equal to one-half of the parameter p (using the same notation as before).

Hence, after taking the value of p from the previous Article:

$$H = w p = \frac{w l^2}{2(\sqrt{f_1} + \sqrt{f_2})^2} = \frac{w l^2}{2(f_1 + 2\sqrt{f_1 f_2} + f_2)} \quad . \quad 11.$$

Let R be the resultant tension at any point, then by the triangle of forces $G N P$, in the figure:

$$R = H \sec i = H \sqrt{1 + \frac{4y^2}{x^2}} \quad . \quad . \quad . \quad . \quad 12.$$

Eq. 12 gives the tension at any point. At the tops of the towers there are found:

$$R_1 = H \sqrt{1 + \frac{4f_1^2}{x_1^2}} \quad . \quad . \quad . \quad . \quad 13.$$

$$R_2 = H \sqrt{1 + \frac{4f_2^2}{x_2^2}} \quad . \quad . \quad . \quad . \quad 14.$$

If $h_1 = h_2$, consequently $x_1 = x_2 = \frac{l}{2}$:

$$H = \frac{w l^2}{8 f}\ *, \quad R_1 = R_2 = H\sqrt{1 + \frac{16 f^2}{l^2}} \quad . \quad . \quad 15.$$

Art. 5.—Length of Curve between Vertex and any Point whose Co-ordinates are x and y, or at which the Inclination to a Horizontal Line is i.

The usual expression for the length of a part of one branch of a parabola, beginning at the vertex, as determined by integral calculus, may easily be put in the following form, c being the length in question:

$$c = \frac{x^2}{4y}\left\{\frac{2y}{x}\sqrt{1 + \frac{4y^2}{x^2}} + \log.\left(\frac{2y}{x} + \sqrt{1 + \frac{4y^2}{x^2}}\right)\right\} \quad 16.$$

Or, using the values for *tan i*, *sec i*, and p, determined in the preceding Articles:

$$c = \frac{p}{2}\left\{tan\, i\, sec\, i + \log.\,(tan\, i + sec\, i)\right\} \quad . \quad . \quad 17.$$

The total length of the cable will be found by putting x_1 and f_1 for x and y in Eq. 16, or i_1 for i in Eq. 17; then x_2 and f_2 for x and y, or i_2 for i, and adding the results.

Representing those results by c_1 and c_2 the total length will then be:

$$c_1 + c_2.$$

By placing $x = \frac{l}{2}$ and $y = f$ in Eq. 16, and taking the towers of equal height, the total length L of the cable between the tops of towers becomes:

$$L = l\frac{l}{8f}\left\{\frac{4f}{l}\sqrt{1+\frac{16f^2}{l^2}} + \log.\left(\frac{4f}{l} + \sqrt{1+\frac{16f^2}{l^2}}\right)\right\} = lZ. \quad 17a.$$

The ratio $\frac{f}{l}$ will generally be found between .2 and .05.

---

* This is the same as the centre flange stress of a beam of span l and depth f carrying the uniform load w per unit of span.

Hence by taking a varying value of that ratio between the above limits and remembering that the natural logarithm is indicated by "log.," the following values of Z may at once be computed:

$f \div l$	Z
.2	1.098223
.175	1.076473
.15	1.056934
.125	1.040194
.1	1.02602
.075	1.014753
.05	1.00662

These values of Z will be found convenient for quick computation of the length of cable in ordinary cases.

An approximate formula sometimes used is determined as follows. In the figure of the first Article of this Chapter, consider $H'P$ to be the arc of a circle, and let x and y be taken as heretofore; also let R be the radius of the circle. The ordinary expression for the length of a circular arc is:

$$\int \frac{dx}{\left(1 - \frac{x^2}{R^2}\right)^{1/2}} = \int \frac{dx}{1 - \frac{x^2}{2R^2}} \text{ (nearly),}$$

if x is small compared with R. Again, performing the division indicated and omitting all terms in the quotient after the second, there will result:

$$\int_o^x dx \left(1 + \frac{x^2}{2R^2}\right) = x\left(1 + \frac{x^2}{6R^2}\right) \quad . \quad . \quad . \quad . \quad 18.$$

If y^2 be omitted in the expression $x^2 = 2Ry - y^2$, and the resulting value of R be inserted in Eq. 18, there will at once be found:

$$x\left(1 + \frac{2y^2}{3x^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 19.$$

As before, to find the total length of the curve, x_1 and f_1, and x_2 and f_2 must be inserted in succession in Eq. 19, and the results added.

If the heights of the towers are equal to each other and to f, the total length will be:

$$l\left(1+\frac{8f^2}{3l^2}\right)^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 20.$$

The expressions 19 and 20 are evidently not close approximations except for flat curves, in which case the nature of the curve is a matter of indifference.

Art. 6.—Deflection of Cable for Change in Length, the Span Remaining the Same.

The approximate formula 19 of the preceding Article may be used to determine the deflection. Approximate formulæ somewhat closer than those given here may be found in Art. 35.

The total length of the cable is:

$$c_1 + c_2 = x_1 + x_2 + \frac{2}{3}\left(\frac{f_1^2}{x_1}+\frac{f_2^2}{x_2}\right),$$

differentiating:

$$d\,(c_1 + c_2) = \frac{4}{3}\left(\frac{f_1}{x_1}+\frac{f_2}{x_2}\right) df\dagger \quad . \quad . \quad . \quad . \quad 21.$$

$$\therefore df = \frac{3\,d\,(c_1 + c_2)}{4\left(\frac{f_1}{x_1}+\frac{f_2}{x_2}\right)} \quad . \quad . \quad . \quad . \quad . \quad 22.$$

The variation in the length of the cable, whether arising from variation in temperature or any other cause, is to be put for $d\,(c_1 + c_2)$ in Eqs. 21 and 22, then df will be the corresponding deflection of the lowest point of the cable.

If the towers are of the same height and, consequently, $c_1 = c_2,\ f_1 = f_2,\ x_1 = x_2 = \frac{l}{2}$:

$$2\,d\,c_1 = \frac{16}{3}\frac{f_1}{l}.\,df. \quad . \quad . \quad . \quad . \quad . \quad . \quad 23.$$

---

* Also see Art. 31, where a closer approximation is made.

† Since $f_1 - f_2 =$ constant, $df_1 = df_2 = df$.

$$d f = \frac{3 l}{16 f_1} \cdot 2 d c_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad 24.$$

It is assumed in Eqs. 21 and 22 that the lowest point of the cable remains at the same horizontal distance from the ends of the span, although that assumption is not strictly accurate.

The true deflection can only be found by trial by the use of Eq. 16 of the preceding Article.

Let $(c_1 + c_2)$ be the known length of the cable before variation in its length takes place; then let f_1, f_2, x_1 and x_2 be the original heights of towers and segments of span, also known. Let y_1 and y_2 be the heights of towers above the lowest point of the cable after the variation in its length has taken place; and let it be assumed, as before, that x_1 and x_2 remain the same whatever the deflection.

Let v be the variation in length of the cable.

Then, since $v = -(c_1 + c_2) + (c_1 + c_2 + v)$:

$$v = \frac{x_1^2}{4 y_1} \left\{ \frac{2 y_1}{x_1} \sqrt{1 + \frac{4 y_1^2}{x_1^2}} + \log. \left(\frac{2 y_1}{x_1} + \sqrt{1 + \frac{4 y_1^2}{x_1^2}} \right) \right\}$$

$$+ \frac{x_2^2}{4 y_2} \left\{ \frac{2 y_2}{x_2} \sqrt{1 + \frac{4 y_2^2}{x_2^2}} + \log. \left(\frac{2 y_2}{x_2} \right. \right.$$

$$\left. \left. + \sqrt{1 + \frac{4 y_2^2}{x_2^2}} \right) \right\} - (c_1 + c_2) \quad . \quad . \quad . \quad 25.$$

But there is also the equation of condition:

$$y_1 - y_2 = f_1 - f_2 = \text{constant} \quad . \quad . \quad . \quad 26.$$

The value of y_1 or y_2 may be taken from Eq. 26 and put in Eq. 25, there will then be but one unknown quantity in the right member of that equation, and its value must be found by trial. The first value of y_1 or y_2 taken may be f_1 or f_2 increased or decreased, as the case may be, by $d f$ given by Eq. 22.

The deflection sought is, of course:

$$y_1 - f_1 = y_2 - f_2.$$

If the new heights, y_1 and y_2, are given, the variation of length, v, will be at once given by Eq. 25.

If heights of towers are the same, f_1, Eq. 26, will not be needed; for making $x_1 = x_2 = \frac{l}{2}$, $c_1 = c_2$, and $y_1 = y_2 = f$, there results:

$$v = \frac{2l^2}{16h}\left\{\frac{4f}{l}\sqrt{1+\frac{16f^2}{l^2}} + \log.\left(\frac{4f}{l} + \sqrt{1+\frac{16f^2}{l^2}}\right)\right\} - 2c_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 27.$$

In Eq. 27, f is to be found by trial, as before, if v is given; or if f is given, v at once results.

The deflection of the middle point of the truss will be:

$$f - f_1.$$

It is to be noticed that in Eqs. 25 and 27 all the quantities, y_1, y_2, and f, increase in the same direction with v. This materially simplifies the approximation by trial.

The determination of v in Eq. 25 might be made without assuming x_1 and x_2 to remain constant, for there are two other equations of condition:

$$x_1' + x_2' = l,$$

and

$$\frac{x_1'^2}{y_1} = \frac{x_2'^2}{y_2}.$$

These, with Eqs. 25 and 26, would be sufficient to find the four unknown quantities, y_1, y_2, x_1' and x_2'.

Such a degree of extreme accuracy, however, is seldom necessary.

Art. 7.—Suspension Cantilevers.

In the figure, $A\,B\,D$ represents a suspension cantilever. The cable $B\,C$ goes over either to another span or to an anchorage, while A is the end of the cantilever. The cable $A\,B$ is in precisely the same condition as the half of a cable belonging to a span equal to $2\,A\,D$; consequently its tension R at any point and its inclination at the same point are to be found by the

formulæ already given. In fact, all the conditions are unchanged except that the platform is subjected to a thrust, uniform

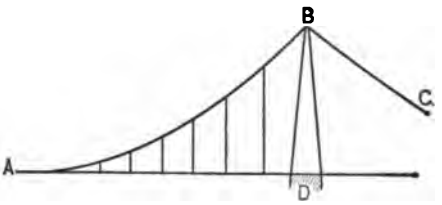

FIG. 2.

throughout its whole length, and equal to the constant horizontal component, H, of the tension R.

Art. 8.—Suspension Bridge with Inclined Suspension Rods—Inclination of Cable to a Horizontal Line—Cable Tension—Direct Stress on Platform—Length of Cable.

In this case the suspension rods or suspenders are all supposed to be parallel and equally inclined to a vertical or horizontal line; they are also supposed to take hold of the platform or stiffening-truss at points equidistant from each other. These conditions cause the cable to be subjected, in each of its parts, to the action of parallel loading of uniform intensity in reference to the span. As was shown in the first Article of this Chapter, the curve of the cable will be composed of common parabolas having axes parallel to the suspension rods.

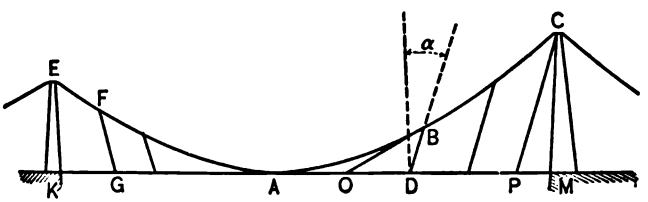

FIG. 3.

In the figure, A is the lowest point of the cable, while $B\,D$ and $F\,G$ represent suspension rods on either side of A. The

angle α is the common inclination of all suspenders to a vertical line; it also represents the inclination of the axis of either of the parabolas $A\,C$ or $A\,E$ to the same line.

The vertex of the parabola $A\,C$ is on the left of A, and the vertex of $A\,E$ is on the right of the same point.

Let x be horizontal in direction and measured from A, and let the length of any suspender, as $B\,D$, be denoted by y, but let $C\,P$ be designated by y_1. Either parabola, as $A\,C$, will then be referred to oblique co-ordinates in the usual manner.

If $O\,B$ be drawn tangent to the curve at B, $A\,O$ will be equal to $O\,D$, or $\frac{x}{2}$. If i represents the inclination of the curve at any point, as B, to a horizontal line, the triangle $O\,B\,D$ will give:

$$\frac{B\,D}{O\,D} = \frac{2\,y}{x} = \frac{\sin i}{\cos\,(\alpha + i)} = \frac{\sin i}{\cos\alpha\cos i - \sin\alpha\sin i}$$

$$\therefore \tan i = \frac{\frac{2\,y}{x}\cos\alpha}{1 + \frac{2\,y}{x}\sin\alpha} \quad . \quad . \quad . \quad 28.$$

In the usual manner, $\sec i = \sqrt{1 + \tan^2 i}$, or,

$$\sec i = \frac{\sqrt{1 + \frac{4\,y}{x}\sin\alpha + \frac{4\,y^2}{x^2}}}{1 + \frac{2\,y}{x}\sin\alpha} \quad . \quad . \quad 29.$$

At the point C, if $C\,M = f_1$, $A\,P = x_1$, and $A\,M = a$, in Eqs. 28 and 29; $f_1 \sec\alpha$ is to be put for y, and $(a - f_1\tan\alpha)$ for x.

Exactly similar equations apply to the other portion of the span.

For the point A, Eqs. 28 and 29 apparently become indeterminate, but only apparently, for the relation,

$$\frac{y}{x^2} = \frac{f_1\sec\alpha}{(a - f_1\tan\alpha)^2} \quad . \quad . \quad . \quad . \quad . \quad 30.$$

gives:

$$\frac{2\,y}{x} = \frac{2\,f_1\sec\alpha}{(a - f_1\tan\alpha)^2}\,x,$$

and when $x = 0$, consequently, $\frac{2\,y}{x}$ becomes zero, making $\tan i$ equal to zero also.

If $O\,B\,D$ be taken as a triangle of forces, $O\,B$ will be the cable tension R at B; while $O\,D$ will be the horizontal component H, and $B\,D$ will represent $w\,x\ sec\ \alpha$. w is the total load per unit of span.

From the triangle in question,

$$\frac{H}{w\,x\ sec\ \alpha} = \frac{cos\ (\alpha + i)}{sin\ i} = \frac{x}{2\,y}, \quad \therefore H = \frac{w\,x^2\ sec\,\alpha}{2\,y},$$

$$\therefore H = \frac{w\,(a - f_1\ tan\ \alpha)^2}{2\,f_1} \quad . \quad . \quad . \quad . \quad . \quad 31.$$

As was to be expected, Eq. 31 shows H to be a constant quantity, but it is not a rectangular component in this case.

The equation of the curve $A\,C$ can now be written by the aid of the equations preceding Eq. 31:

$$x^2 = \frac{(a - f_1\ tan\ \alpha)^2}{f_1\ sec\ \alpha}\ y = \frac{2\,H}{w\ sec\ \alpha}\ y \quad . \quad . \quad . \quad 32.$$

This equation is evidently equally applicable to the curve $A\,E$.

The same triangle gives for the resultant tension at any point:

$$R = \sqrt{(wx\ sec\ \alpha)^2 + H + 2^2 wx\ H\ tan\ \alpha} \quad . \quad . \quad 33.$$

For the point C, x becomes $(a - f_1\ tan\ \alpha)$.

If l is the span, these equations apply to the other portion of it, by taking f_2 for f_1 and $(l - a)$ for a.

If the towers are of equal heights, h_1 becomes equal to h_2, and $a = l - a = \frac{l}{2}$.

Let p be the horizontal distance between any two suspenders, then the tension, t, in the suspender will be:

$$t = w\,p\ sec\ \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad 34.$$

The direct stress in the platform is caused by the horizontal component of the tension in the suspension rods. This stress may exist as tension in the platform, in which case it will exert no action on the towers. Remembering that all the suspension

rods must, at any instant, be subjected to a uniform stress, it is evident that the direct tension in the platform will have its greatest value at the centre, and will be equal to·

$$n\,t\,\sin\alpha = n\,w\,p\,\tan\alpha;$$

in which n is the number of suspension rods in each half of the span, supposing towers to be of equal heights. If n' be the number of suspenders between the end of the span and any point, the tension in the platform at that point will be:

$$n'\,t\,\sin\alpha = n'\,w\,p\,\tan\alpha.$$

If the towers are of unequal heights, there will be a greater number of suspenders on one side of the lowest point of the cable than on the other. Let n_1 be the number in that portion of the span adjacent to the highest tower, and n_2 the number in the other portion; n_1 will be greater than n_2. In this case, then, the platform at the foot of the highest tower will sustain a thrust given by the expression:

$$(n_1 - n_2)\,t\,\sin\alpha = (n_1 - n_2)\,w\,p\,\tan\alpha.$$

If the platform is to sustain a direct thrust only, at the feet of the two towers it will have to sustain thrusts given by the expressions:

$$n_1\,t\,\sin\alpha = n_1\,w\,p\,\tan\alpha$$
$$n_2\,t\,\sin\alpha = n_2\,w\,p\,\tan\alpha.$$

If n' represents the number of suspension rods between the centre and any point, the thrust at that point will be:

$$n'\,t\,\sin\alpha = n'\,w\,p\,\tan\alpha.$$

In the case of a suspension cantilever, in addition to the thrust given above there will be one denoted by H, uniform throughout its length. Other calculations for a suspension cantilever are precisely the same as those already given.

The length of the cable from the lowest point to any other point at which the inclination to a horizontal is i, is readily found by means of the formula used for the cable with vertical rods. In the present case the inclination of the cable at any point to a line perpendicular to the axis of the parabola is $(i + \alpha)$; consequently there is simply to be found the length of the

parabolic arc between the points at which the inclinations to the axis are $(90 - (i + \alpha))$ and $(90 - \alpha)$.

The formula mentioned then gives:

$$c = \frac{p}{2}\left(\tan(i+\alpha)\sec(i+\alpha) - \tan\alpha\sec\alpha + \log.\frac{\tan(i+\alpha)+\sec(i+\alpha)}{\tan\alpha+\sec\alpha}\right) \quad . \quad 35.$$

It is known from analytical geometry that p takes the following form in terms of the oblique co-ordinates used in this case:

$$p = \frac{x^2\cos^2\alpha}{2y} = \frac{(\alpha - f_1\tan\alpha)^2\cos^3\alpha}{2f_1}.$$

Eq. 35 is, of course, to be applied to both branches of the curve to obtain the total length.

From what was said in the demonstration of the approximate formula, it may be seen that it can be applied to the present case by changing x to $(x + y\sin\alpha)$ and y to $y\cos\alpha$. The formula then becomes:

$$c = x + y\sin\alpha + \frac{2y^2\cos^2\alpha}{3(x+y\sin\alpha)} \quad . \quad . \quad . \quad . \quad 36.$$

Art. 9.—Suspension Rods: Lengths and Stresses.

In the following calculations it is virtually assumed that the cable lies in a vertical plane, and that the suspension rods are vertical. This, however, does not affect the generality of the

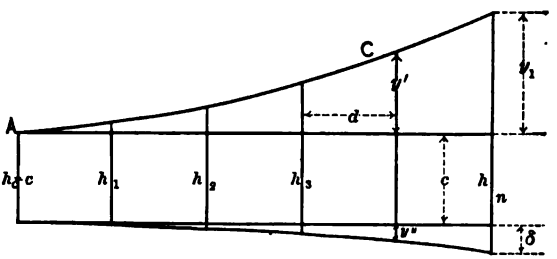

FIG. 4.

results obtained, for in all cases the suspension rods are supposed parallel to each other, and the lengths found by the

formulæ of this Article are to be taken as the vertical projections of the true or actual lengths. The true lengths are therefore to be found by multiplying the values of h_0, h_1, h_2, etc., by the secant of the common inclination of the suspension rods to a vertical line.

Since a flat parabola nearly coincides with a circle, the camber may be supposed to be formed by a parabolic arc. Let the co-ordinate x be measured from A toward B, in Fig. 4, y being perpendicular to it; also let $A\,B = x_1 =$ half span. Then since the curve of the cable is supposed to be a parabola in a vertical plane:

$$y' = y_1 \frac{x^2}{x_1^2}.$$

In the same manner for the camber:

$$y'' = \delta \frac{x^2}{x_1^2}.$$

Then the total length of any suspender is:

$$h = y' + y'' + c.$$

When the suspenders are separated by a constant distance, d, simpler formulæ may be found.

Each suspender is composed of the sum of two variable lengths (y' and y'') and a constant length, c. If $(y_1 + \delta)$ be written for y_1:

$$y = (y_1 + \delta) \frac{x^2}{x_1^2}$$

will evidently be the sum of the two variable lengths referred to. Hence, if h_0, h_1, h_2, h_3, * * * h_n represent the lengths of the suspenders as shown in the figure:

$$h_0 = c,$$

$$h_1 = c + \frac{d^2}{x_1^2}(y_1 + \delta),$$

$$h_2 = c + \frac{4\,d^2}{x_1^2}(y_1 + \delta),$$

$$h_3 = c + \frac{9\,d^2}{x_1^2}(y_1 + \delta),$$

$$h_n = c + \frac{n^2 d^2}{x_1^2}(y_1 + \delta),$$

Or: $$h_n = c + \left(\frac{n^2 d^2}{x_1^2} = 1\right)(y_1 + \delta) = c + y_1 + \delta.$$

Having computed the lengths of the suspenders for one half the span, the results may be used for the other half if the piers are of the same height; otherwise the lengths must be computed separately.

The stress in any suspension rod is the vertical load which it carries multiplied by the secant of its inclination to a vertical line.

Art. 10.—Pressure on the Tower—Stability of the Latter—Anchorage.

Let P_v = vertical component of pressure on tower head.

Let P_h = horizontal component of pressure on tower head.

Let R = resultant pressure on tower head.

Let T_p and T_p', Fig. 5, = tensions of the cable on different sides of the top of the tower.

α, α', and θ represent inclinations to the vertical as shown.

In general the cables at the top of the tower may have

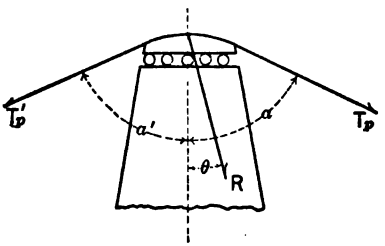

FIG. 5.

different inclinations on the two sides, as shown in Fig. 5. All friction, rolling or other, will first be neglected. Hence:

$$P_v = T_p \cos\alpha + T_p' \cos\alpha'; P_h = T_p \sin\alpha - T_p' \sin\alpha' \quad . \quad 37.$$

$$R = \sqrt{P_v^2 + P_h^2}; \cos\theta = \frac{P_v}{R}. \quad . \quad . \quad . \quad . \quad . \quad 38.$$

If it should be desirable to consider friction under the saddle, *e.g.*, if ϕ be the coefficient of that friction, then:

$$T_p \sin \alpha - T_p' \sin \alpha' \leq \phi P_v \quad . \quad . \quad . \quad . \quad 39.$$

Eq. 39 must be satisfied if the stability of the saddle at the top of the tower be maintained.

If the cable has the same inclination on the two sides of the top of the tower so that $\alpha = \alpha'$, $P_v = 2W$, $P_h = 0$, $R = 2W$, $\theta = 0$. ($W = \frac{1}{2}$ weight of load and structure.)

It is thus seen that if the cable has the same inclination on either side of the tower, the total vertical pressure on the latter is equal to the entire weight of the structure and its load between the two towers.

If the tower is of masonry it is essential that the resultant pressure on it shall be practically vertical. This condition will clearly be attained when the inclination of the cable is the same on each side of the tower, as the horizontal component of the cable stress in the main cable and in the back-stay will then be the same. In general the resultant pressure will be vertical when the horizontal component of the cable stress on either side of the tower is the same as that on the other. If the cable inclinations α and α' are not the same the following condition, written by the aid of Eq. 37, must obtain:

$$P_h = 0, \therefore T_p \sin \alpha = T_p' \sin \alpha' \quad . \quad . \quad . \quad . \quad 40.$$

Obviously there will be some friction at the top of the tower so that the horizontal components of the cable stresses may not be precisely equal, but that friction is in general negligible in this connection.

Steel towers are now usually constructed instead of masonry. These steel towers are in reality long columns, either fixed and anchored at the base, or, as has been proposed, hinged at the base on a pin of large diameter. Their design therefore will involve the consideration of long columns as wholes, as well as long columns and bracing of their separate parts, much like viaduct towers.

When steel towers are fixed and anchored at the base, the friction of the saddle at the top should be considered in its bend-

ing effect upon the tower as a whole, and the resulting stresses in the different members of the tower should be determined accordingly in the design. The effects of the lateral wind pressures must also be recognized, and the resulting stresses must be determined. Inasmuch as the lateral wind pressure against the entire structure will be carried to the towers through the bracing between the stiffening-trusses, the wind stresses in the towers below the stiffening-trusses may be large; at any rate, they need full recognition and careful treatment in the design of the structure.

In discussing the stability of position of masonry towers, let the distance of the centre of pressure from the centre of figure of the section of the pier be denoted by q'. If this latter does not exceed q (the limit of safety for q'), which may be ascertained by determining the line of resistance for the pier, stability of position will be secured.

It is supposed, of course, that θ has some value greater than zero; otherwise $q' = 0$.

The stability of friction for masonry towers will be secured, at any joint, if the obliquity of the resultant pressure be less than the angle of repose.

Anchorage.

If the anchorage is a mass of masonry, the stabilities of position and friction are to be considered.

Let W = weight of the mass, and q' the normal distance from a vertical line through its centre of gravity to the centre of figure of its base. Let T_p equal the tension in the anchor chains or cables, and d the normal distance from the centre of pressure to its line of direction; and q the greatest distance from the centre of pressure of the base of the foundation to its centre of figure. The relations of these quantities is fully shown in Fig. 6. The anchorage should be so designed as to make the lever arm of W the sum of q' and q, rather than their difference.

Then, in order that stability of position may be secured:

$$T_p d \angle W (q + q').$$

The centre of pressure, P, on the foundation bed must not be farther from F, the centre of figure of that bed, than is permissible in view of the greatest intensity of foundation pressure at E. If the foundation bed is rectangular in form, it is considered safe and proper to make $P\,E$ never less than $\frac{1}{3} \times A\,E$, *i. e.*, $F\,P$ never greater than $\frac{1}{6} \times A\,E$. In this case the intensity of pressure at A can never be less than zero.

Stability of friction is secured if the greatest obliquity of the resultant pressure on any section (including the base) is less than the angle of repose for the surfaces in contact.

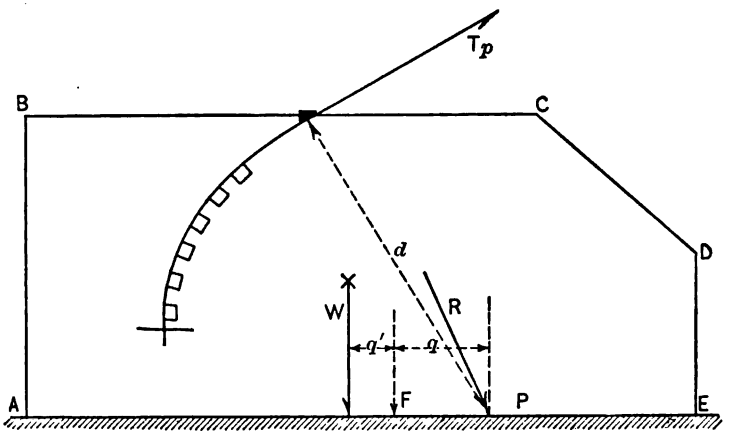

FIG. 6.

If it were not for friction between the anchor chain and its supports, on the circular part of the chain, the tension would be the same throughout its whole length; but on account of friction, the tension diminishes on the circular part, from link to link downward, according to the law of friction between cords and cylinders, and is, therefore, the least at the bottom.

The diminution of tension of the anchor chain is computed by the formula:

$$T_p = T_p' e^{\phi\theta}$$

in which T_p = tension of anchor chain before friction takes effect; T_p' = tension of any point below the first point of

support; e = base of *Napierian system of logarithms;* ϕ = coefficient of friction; θ = length of arc considered. The anchorage can be ruptured only by the breaking of the chain, or plate, or by pulling out the whole masonry. The probability of the latter can be determined by comparing the tension of the chain, at the upper surface of the masonry, with the weight of the whole masonry.

Art. 11.—Theory of the Stiffening-Truss—Ends Anchored—Continuous Load—Single Weight.

It has been seen that when a perfectly flexible suspension-bridge cable carries a load of uniform intensity over the entire span, its centre line forms a parabolic curve. When, however, such a cable carries an isolated weight or a partially uniform load, whether moving or fixed, it is evident that the centre line of the cable will assume a form different from that of the parabola, thus causing objectionable undulations, unless such a change is prevented by some special device. Such a special device is the stiffening-truss.

Two cases may arise: the stiffening-truss may be securely anchored at its ends; or its ends may simply rest upon supports and be free to rise, in which case there can be no negative or *downward* reaction.

The former case will be considered first.

The cable must retain the same parabolic form for all positions of the moving load. It has already been seen that such a result can be attained only by causing a uniform pull on the suspension rods from end to end of the span. Let T be the general expression for this uniform pull for any suspension rod, and let t be its intensity per unit of span, so that if p be the panel length of the stiffening-truss, $T = p\,t$. Let w be the weight per unit of span of the fixed load sustained by the cables. This will be composed of the weights of the truss, suspension rods and cable or cables. Let w' be the moving load per unit of span; l the span; R the reaction at B, Fig. 7; R' the reaction at A, and let the moving load pass on the bridge from B.

Also let x_1 be the distance from B to the head of the moving load; the latter being supposed continuous from B.

Since all the forces acting on the stiffening-truss are vertically in direction, there are only two general conditional equations

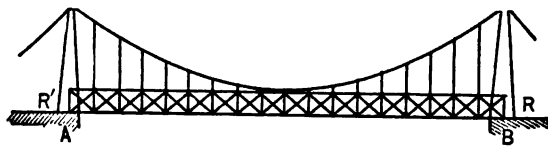

FIG. 7.

of equilibrium, and those simply indicate that the sum of all the external vertical forces must be zero as well as the sum of their moments about any point.

Those two equations are the following:

$$w\,l + w'\,x_1 - t\,l - R - R' = 0 \quad . \quad . \quad . \quad 41.$$

$$(w + w')\frac{x_1^2}{2} - t\frac{x_1^2}{2} - R\,x_1 + (t - w)\frac{(l - x_1)^2}{2} + R'\,(l - x_1) = 0. \quad . \quad 42.$$

Eq. 42 can be at once written by taking moments about the point x_1 at the head of the moving load.

Eqs. 41 and 42 are the only equations of condition necessary for equilibrium, but they hold three unknown quantities, *i. e.*, t, R, and R'; hence, any one of those three quantities may be assumed at pleasure, and the other two determined from Eqs. 41 and 42. This indetermination simply means that unless another condition be imposed, it cannot be ascertained how much the truss will carry *as a simple truss*, and how much in connection with the cable, *i. e.*, the stress conditions in the truss are statically indeterminate.

This other condition is virtually the following: *the stiffening-truss must act wholly in connection with the cable, and carry no load whatever as an ordinary truss.*

The direct consequence of this condition of the problem is that the sum of all the uniform upward forces $T = p\,t$ must be equal in amount to the sum of all the loads of the kinds

w and w'. The line of action of the resultant of the latter is not, for a partial moving load, the line of action of the resultant of the former; consequently the truss will be subjected to the action of a couple. In order that equilibrium may be assured, therefore, another couple of equal moment but opposite sign must be applied to the truss; the forces of this couple must act at the extremities A and B, and they are the reactions R and R'. From this there at once results:

$$R = -R'.$$

This condition gives in Eq. 41:

$$t = w + w' \frac{x_1}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 43.$$

Eq. 42 gives:

$$R = -R' = \frac{w' x_1}{2} \left(1 - \frac{x_1}{l}\right) \quad . \quad . \quad . \quad . \quad 44.$$

Eq. 44 shows that both reactions R and R' are zero for $x_1 = l$, or for $w' = 0$.

It is also seen *that R and R' are always numerically equal, but have opposite directions, hence R' is a downward reaction, and its maximum value will indicate the amount of anchorage required at each end of the truss.*

Using Eq. 44:

$$\frac{dR}{dx_1} = \frac{w'}{2} - \frac{w' x_1}{2l} - \frac{w' x_1}{2l} = 0, \qquad \therefore x_1 = \frac{l}{2}.$$

Putting $x_1 = \frac{l}{2}$ in Eq. 44:

$$R = \frac{w'l}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 45.$$

Eq. 45 shows the maximum value of $-R'$ and gives the amount of anchorage required at either end of the truss; it also shows the greatest shear to be provided for at either end of the truss.

The same results can be more obviously shown graphically as by Fig. 8, in which the span is represented by $A\,B$. Eq. 44

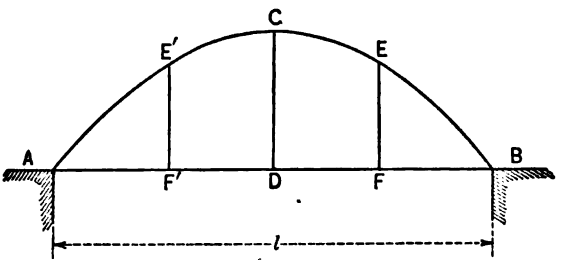

FIG. 8.

is the equation of a parabola with a vertical axis $C\,D$, the origin being on the curve at B.

The vertical ordinates, as $E\,F$, $C\,D$ and $E'\,F'$, represent the reactions $R = -R'$ where the load covers $B\,F$, $B\,D$ and $B\,F'$, respectively. Evidently the maximum reaction is the central ordinate $C\,D$ for $x_1 = \frac{l}{2} = B\,D$, which has the value $R = \frac{w'l}{8}$ by Eq. 44.

The general value for the shear at any section of the portion of the truss covered by the moving load is:

$$S = R + t\,x - w\,x - w'\,x \quad . \quad . \quad . \quad . \quad 46.$$

Or,

$$S = w'\left(1 - \frac{x_1}{l}\right)\left(\frac{x_1}{2} - x\right) \quad . \quad . \quad . \quad . \quad 47.$$

This value of S shows it to be positive near the end of the bridge; it then decreases as x increases, passes through the value zero, for $x = \frac{x_1}{2}$, and then increases as a negative quantity. As a negative quantity it attains its maximum value for $x = x_1$; it then becomes:

$$S_1 = -\frac{w'\,x_1}{2}\left(1 - \frac{x_1}{l}\right) = R' \quad . \quad . \quad . \quad 48.$$

Hence the two reactions at the ends of the truss and the shear at the head of the moving load are always numerically equal. Eq. 48, consequently, takes its maximum value for $x_1 = \frac{l}{2}$, and that value is given in Eq. 45; this last equation, therefore, gives the maximum shear which is to be provided for at the head of the moving load.

Since this maximum shear is to be provided for at both ends and at the middle of the truss, it might be advisable in some cases to design all the web members of the truss to be of uniform size, and capable of carrying this maximum shear, although there would then be a little waste of material in the vicinity of the quarter points of the span on each side of the centre.

This supposes, of course, that the chords of the stiffening-truss are parallel and horizontal. If the chords are not parallel the amount of shear carried by the web members will depend on the inclination of one or both the chords.

For all values of x and x_1, for the portion of the span covered by the moving load, the total shear will be given by Eq. 47.

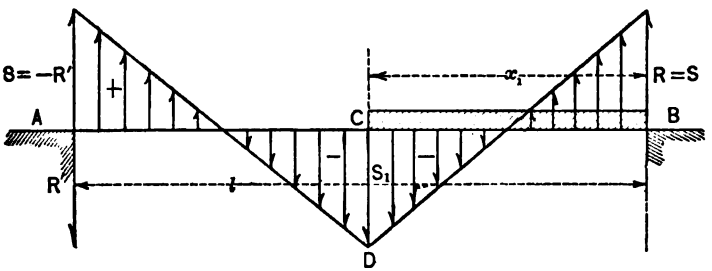

FIG. 9.

For the portion of the span not covered by the moving load the general value for the shear is (*measuring x from A*):

$$S = -R' + wx - tx = w' x_1 \left(\tfrac{1}{2} - \frac{x_1}{2l} - \frac{x}{l}\right) \quad . \quad 49.$$

This expression is the equation of a straight line and it

attains its greatest values, for $x = 0$ and $x = l - x_1$. In the first case the shear becomes $-R'$, and in the second, R.

These results can be shown graphically by Fig. 9, in which $A\,B$ is the length, l, of the span, and $C\,B$ is any length, x_1, of train. The shear at the head of the train is $C\,D$ given by Eq. 48, and it is equal to $R = -R'$. When $C\,B = \frac{l}{2}$ the shear $C\,D$ is the greatest possible; its value is given by Eq. 45. In the latter case, the points of no shear, F and G, are quarter points.

Since the maximum shear in a simple truss, of the span l and uniform loading of intensity $(w + w')$, is $\frac{1}{2}(w + w')\,l$, it is seen that the maximum shear in the stiffening-truss of same span is only one-fourth of that due to the moving load alone in the case of the simple truss.

The general value of the bending moment to which the truss is subjected, for the portion covered by the moving load, is:

$$M = R\,x - (w + w' - t)\frac{x^2}{2},$$

$$\therefore M = \frac{w'\,x}{2}\left(x_1 - x\right)\left(1 - \frac{x_1}{l}\right) \quad . \quad . \quad . \quad 50.$$

Eq. 50 shows that if $x = x_1$, the bending moment is equal to zero. Hence, *at the head of the moving load, for all its positions, there is a section of contraflexure or no bending, and consequently the loaded and unloaded portions of the stiffening-truss are each in the condition of a simple beam supported only at each end, and loaded uniformly throughout their lengths.*

If $\frac{d\,M}{d\,x}$ be put equal to zero, it will be found that the bending moment has its maximum value for $x = \frac{x_1}{2}$; as might have been anticipated.

Putting $x = \frac{x_1}{2}$ in Eq. 50:

$$M = \frac{w'\,l^2}{8}\,\frac{x_1^2}{l^2}\left(1 - \frac{x_1}{l}\right) \quad . \quad . \quad . \quad . \quad 51.$$

By differentiating in respect to x_1 it will be found that M

has its maximum value, for $x_1 = \frac{2}{3} l$. Denoting this value of M by M_1, there results:

$$M_1 = \frac{w' l^2}{54} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 52.$$

Eq. 52 shows the maximum bending moment to which any loaded portion of the truss can be subjected.

The following table will make the application of Eq. 51 simple and easy. It can also be used to construct a convenient curve. The left hand column for $\frac{x_1}{l}$ is to be used for Eq. 51.

$\frac{x_1}{l}$ For Eq. 51	$\frac{x_1^2}{l^2}\left(1 - \frac{x_1}{l}\right)$	$\frac{x_1}{l}$ For Eq. 54
.05	.002375	.95
.1	.009	.9
.15	.019125	.85
.2	.032	.8
.25	.046875	.75
.3	.063	.7
.35	.079625	.65
.4	.096	.6
.45	.111375	.55
.5	.125	.5
.55	.136125	.45
.6	.144	.4
.65	.147875	.35
.7	.147	.3
.75	.140625	.25
.8	.128	.2
.85	.105375	.15
.9	.081	.1
.95	.045125	.05

If a simple truss supported at each end be subjected to the action of a uniform load, of the intensity $(w + w')$, throughout its entire length, the greatest bending moment will be:

$$M' = \frac{(w + w') l^2}{8} = \frac{w' l^2}{8} \text{ (if } w = 0);$$

$$\therefore M_1 = \tfrac{4}{27} M_0' > \tfrac{1}{7} M_0' \quad . \quad . \quad . \quad 53.*$$

* The subscript 0 indicates that $w = 0$ in M'.

From what has already been shown it is evident that the greatest bending moment for the portion of the truss not covered by the moving load will occur at the distance $\left(\frac{l - x_1}{2}\right)$ from the reaction R'. The general value, therefore, for the greatest moment for that portion will be:

$$M = \tfrac{1}{2} R' (l - x_1) + (t - w) \frac{(l - x_1)^2}{8},$$

$$\therefore M = - \frac{w' l^2}{8} \frac{x_1}{l} \left(1 - \frac{x_1}{l}\right)^2 \quad . \quad . \quad . \quad . \quad 54.$$

Putting the differential coefficient of M in respect to x_1 equal to zero, there results:

$$x_1 = \tfrac{2}{3} l \pm \sqrt{\tfrac{4}{9} l^2 - \tfrac{8}{9} l^2},$$

$$\therefore x_1 = \tfrac{2}{3} l \pm \frac{l}{3} = l \text{ or } \tfrac{1}{3} l \quad . \quad . \quad . \quad . \quad 55.$$

The latter value, $\frac{1}{3} l$, gives a maximum, and inserting it in Eq. 54:

$$M_1' = - \frac{w' l^2}{54} = - M_1 = - \tfrac{1}{7} M_0' \text{ (nearly)} \quad . \quad 56.$$

The preceding table used in connection with Eq. 51 is directly applicable to Eq. 54, as shown in the table itself, by reading the values of $\frac{x_1}{l}$ from the bottom up rather than from the top down.

Eqs. 52 and 56 show that *the greatest bending moments to which the stiffening-truss is subjected are equal, but of opposite kinds; and it is seen that they occur when one-third or two-thirds of the span are covered by the moving load.* The chords, therefore, of the stiffening-truss must be designed to resist both tension and compression produced by the reversal of bending moments due to different positions of that moving load.

Eqs. 50 and 54 are general expressions for all the bending moments to which any portion of the truss can possibly be subjected, but in some cases it may be best to make the chords

uniform in section (supposing the depth of the truss to be constant) from end to end, and capable of resisting the moments given by Eqs. 52 and 56.

The general condition of bending in the stiffening-truss is shown by Fig. 10. The uniform moving load covers $B\,C = x_1$ while $A\,C = l - x$ is free from that load. When the positive maximum bending moment exists in $B\,C$, $x_1 = \frac{2}{3}\,l$; but when the negative maximum moment exists in $A\,C$, $x_1 = \frac{1}{3}\,l$. It must be remembered that the moving load may pass in either direction.

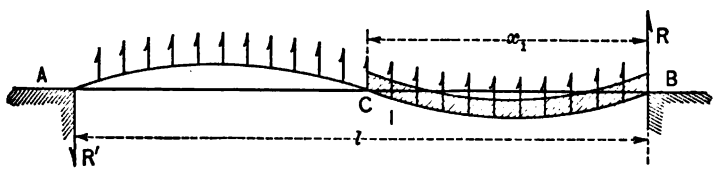

FIG. 10.

Eq. 53 shows that the greatest bending moment to which a stiffening-truss can be subjected is about one-seventh of that found in a simple truss supported at each end and loaded with a uniform load equal in intensity to that of the moving load on the stiffening-truss.

It may be interesting to notice that the resultant load on the portion x_1 of the truss is *downward*, since $(w + w') > \left(t = w + \frac{w'\,x_1}{l}\right)$, while that on the portion $(l - x_1)$ is *upward*, since $w < t$.

Single Weight.

If the bridge is traversed by a single concentrated load or weight, W, the general method of procedure is precisely the same as before. Let the weight W pass on the bridge from the end B in Fig. 11, and let x_1 denote its distance from that point; also measure x from the same point.

The general equations of condition are:

$$R + R' + t\,l - w\,l - W = 0 \quad . \quad . \quad . \quad . \quad 57.$$

$$-R'(l - x_1) + (w - t)\frac{(l - x_1)^2}{2} - (w - t)\frac{x_1^2}{2} + R\,x_1 = 0.$$

$$R'\,l + (t - w)\frac{l^2}{2} - W\,x_1 = 0 \quad . \quad . \quad . \quad . \quad 58.$$

Eq. 58 is written at once by taking moments about the point B, Fig. 11.

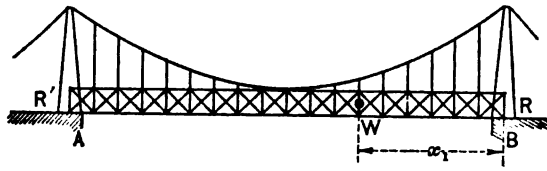

FIG. 11.

By precisely the same method as before, there may be found the result, $R = -R'$.

In Eqs. 57 and 58 let there be put $R = -R'$, then there results:

$$t = w + \frac{W}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 59.$$

$$R = -R' = W\left(\tfrac{1}{2} - \frac{x_1}{l}\right) \quad . \quad . \quad . \quad . \quad 60.$$

When $x_1 = \frac{l}{2}$, $R = 0$ and $R' = 0$.

Eq. 60 shows that *the reaction nearest the weight W will always be positive, or upward; and that the other will be negative, or downward.*

The maximum value of R or R' is found (by making $x_1 = 0$ in Eq. 60) to be $\frac{W}{2}$, and *that is the amount of anchorage required for the weight W, alone, at each end of the truss.*

The value of the reaction $R = -R'$, for any position of W can readily be found graphically by Fig. 12. The length of span l is represented by $A\,B$, and C is at its centre. If $B\,D = A\,F$ is laid off to represent $\frac{W}{2}$ by scale, and if the straight line

$D\,C\,F$ is drawn, then $H\,K$ will be the value of R if W is at H. Similarly $L\,N$ will be the reaction $-R$, that reaction being negative when x_1 is greater than $\frac{l}{2}$. Any ordinate between $D\,F$ and $A\,B$ drawn at any point will represent the reaction R when W is at that point. The line $D\,C\,F$ is therefore the influence line for the reactions R and $-R$.

The point of application of the weight W divides the span into two segments. The general value of the shear for the shorter segment is:

$$S = t\,x - w\,x + R = \frac{W}{l}\,x + W\left(\tfrac{1}{2} - \frac{x_1}{l}\right) \quad . \quad . \quad 61.$$

This has its greatest value for $x = x_1$; it then becomes:

$$S' = \frac{W}{2}.$$

Hence S' is the uniform maximum upward shear that must be provided for throughout the whole length of the truss.

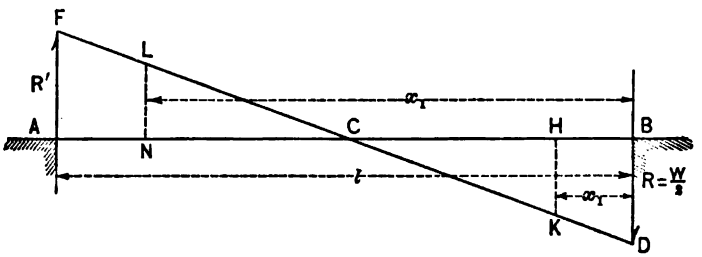

FIG. 12.

The general value of the shear in the longer segment of the span is:

$$S = R' - w\,(l - x) + t\,(l - x) = W\left(\tfrac{1}{2} + \frac{x_1}{l} - \frac{x}{l}\right) \quad . \quad 62.$$

This expression attains a positive maximum for $x = x_1$, that being the least positive value of x admissible; the resulting value of S is $\frac{1}{2}\,W$, which shows nothing new.

The negative maximum for $x = l$ is simply the reaction R'. Putting $S = 0$ in Eq. 62, there results:

$$x = x_1 + \frac{l}{2}.$$

Hence for all points of the span between $\frac{l}{2} + x_1$ and l there will be negative or downward shear. The maximum negative shear, however, to be provided for, is shown by Eq. 62 to exist in one-half of the truss *when W rests on the opposite end;* or, when in that equation $x_1 = 0$ and $x > \frac{l}{2}$. The negative or downward shear to be provided for has, then, for its general expression:

$$S_1 = -W\left(\frac{x}{l} - \tfrac{1}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad 63.$$

in which equation $x > \frac{l}{2}$.

Eq. 63 is to be applied to each half of the truss, and it is also seen that the web members which take up S_1 should increase, in ultimate resistance, uniformly from the centre to the ends of the truss; supposing the chords to be parallel and horizontal. It may be best to design them of uniform dimensions belonging to those at the ends.

These considerations show that the ordinates, as $H K$ and $L N$, in Fig. 12, drawn between $D F$ and $A B$, also represent the shears at the point at which they are drawn.

The bending moment for any point of the smaller segment of the truss is:

$$M = R x + (t - w)\frac{x^2}{2} = W\left(\tfrac{1}{2} - \frac{x_1}{l}\right)x + \frac{W x^2}{2 l} \quad . \quad 64.$$

Since both terms of this moment are positive, it will attain its greatest value, for $x = x_1$; it then becomes:

$$M_1 = \frac{W}{2}\left(x_1 - \frac{x_1^2}{l}\right) = \frac{W l}{2}\frac{x_1}{l}\left(1 - \frac{x_1}{l}\right) \quad . \quad . \quad . \quad 65.$$

This equation shows that the positive bending moment at

the location of W is the same for the segments of span x_1 and $(l - x_1)$. Hence Eq. 65 gives the general value of the greatest positive bending moment to which any point of the truss will be subjected, *for which case x_1 need never be made greater than $\frac{1}{2} l$.*

The bending moment M_1 will evidently cause compression in the upper chord, and tension in the lower.

The table given in connection with Eq. 76 is directly available for Eq. 65.

For the longer segment of the truss the general value of the bending moment is:

$$M = R'(l - x) + (t - w)\frac{(l - x)^2}{2}.$$

Or,
$$M = \frac{W l}{2}\left(1 - \frac{x}{l}\right)\left(\frac{2 x_1}{l} - \frac{x}{l}\right) \quad . \quad . \quad . \quad 66.$$

There is evidently a point of contraflexure for the longer segment of the truss, for if the second member of Eq. 66 be put equal to zero, there results $x = 2 x_1$. Hence *the portion $l - 2 x_1$, of the truss, will be subjected to a negative bending moment causing tension in the upper chord and compression in the lower.*

The general condition of the stiffening-truss is shown by Fig. 13, in which $A B = l$ is the length of span and x_1 is the distance

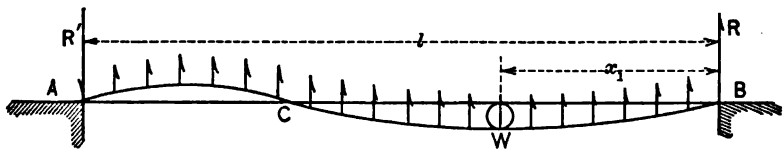

FIG. 13.

of the load W from the end B of the span. C is the point of no bending or contraflexure and, hence, $B C = 2 x_1$. The upward pull of the suspenders is shown by the arrows. It is clear that the part $C B$ of the stiffening-truss is subjected to positive bending, while $A C$ is subjected to negative bending.

In Eq. 66, putting $\frac{d M}{d x} = 0$, there results:

$$x = x_1 + \frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 67.$$

This value of x, in Eq. 66, gives:

$$M' = -\frac{W l}{2}\left(\tfrac{1}{2} - \frac{x_1}{l}\right)^2 \quad . \quad . \quad . \quad . \quad 68.$$

Eq. 67 shows that x must always be greater than $\frac{l}{2}$ for the greatest negative moment.

The following table will be found useful in applying Eq. 68.

$\frac{x_1}{l}$	$\frac{1}{2} - \frac{x_1}{l}$	$(\frac{1}{2} - \frac{x_1}{l})^2$
.05	.45	.2025
.1	.4	.16
.15	.35	.1225
.2	.3	.09
.25	.25	.0625
.3	.2	.04
.35	.15	.0225
.4	.1	.01
.45	.05	.0025
.5	.0	.0

Eq. 68 *gives the general value for the maximum negative bending moment at any point in the entire truss, for any position of the weight* W. In Eq. 68 it is to be remembered, x_1 must always be less than $\frac{l}{2}$; also, that the point at which M' exists will be given by the value of x in Eq. 67.

Eqs. 65 and 68 show that the greatest positive and negative bending moments occur at the centre of span and that they have the value:

$$M_1 = -M' = \frac{W l}{8} \quad . \quad . \quad . \quad . \quad . \quad 69.$$

These results are all shown graphically by Fig. 14. In that figure, $A\,B$ is the length of span l, C being at its centre. $F\,C$ $(= C\,G)$ is the greatest positive (centre) moment $\frac{W l}{8}$ in the stiffening-truss, $C\,G$ being the greatest negative moment $-\frac{Wl}{8}$, also at the centre of span. The curve $F\,B$ is a parabola

with its vertex at F, and it represents Eq. 65. The greatest positive moment at J is $I\,J$, and similarly for other points.

In the same manner, the parabolic curve $A\,G$, with vertex at A, represents Eq. 68. The greatest negative moment at K, located by Eq. 67, is $K\,L$ given by Eq. 68. Any other vertical ordinate between $A\,C$ and $A\,G$ will be the greatest negative moment at the point at which it is drawn. As this parabola $F\,B$ is identical with $A\,G$, the negative ordinates of the kind $K\,L$ are the same as the ordinates $K'\,L'$ between the horizontal

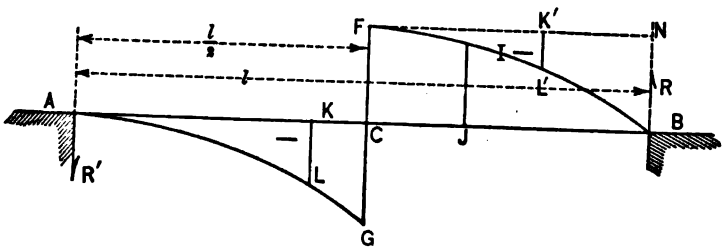

FIG. 14.

line $F\,N$ and $F\,B$, $K'\,L'$ being the same distance from $N\,B$ that $K\,L$ is from $C\,G$. The lines $B\,I\,F$ and $A\,L\,G$ are influence lines for the greatest positive and negative bending moments.

The formulæ for a continuous load, taken in connection with those for a single weight, will give all the circumstances of bending or shearing which can exist with any condition or position of loading.

When the shear has been determined for any section, the stress in the web member which is to carry it (if the chords are parallel and horizontal) will be found at once by multiplying that shear by the secant of inclination of the web member to a vertical line. If there are two or more systems of triangulation in the truss, each system may be approximately treated as a single truss in the usual manner.

If desirable, after the reactions R and R' and the upward load $T = p\,t$ are known, the stresses in the individual members of the stiffening-truss can be treated as in the case of an ordinary truss supported at each end.

Art. 12.—Theory of the Stiffening-Truss—Ends Free—Continuous Load—Single Weight.

In this Article the notation of the previous one will be continued, and the same figure will be referred to.

The case of a continuous load will first be treated, and, as before, it will be supposed to pass on the bridge from B.

Since the ends are not anchored, in this case there can be no negative or downward reaction, *consequently R' will be zero.*

As before, putting the sum of all the vertical forces acting on the truss equal to zero, and taking moments about the head of the moving load, there result the two general Equations of condition:

$$w\,l + w'\,x_1 - t\,l - R = 0 \quad . \quad . \quad . \quad 70.$$

$$(w + w' - t)\,\frac{x_1^2}{2} - (w - t)\,\frac{(l - x_1)^2}{2} - R\,x_1 = 0 \quad . \quad 71.$$

Since there are now but two unknown quantities, R and t, the problem is perfectly determinate. Eqs. 1 and 2 give:

$$R = w'\,x_1\left(1 - \frac{x_1}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad 72.$$

$$t = w + \frac{w'\,x_1^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad 73.$$

The general value for the shear at any section of the truss, for the portion covered by the moving load, is

$$S = R + (t - w - w')\,x = w'\,(x_1 - x) - w'\,\frac{x_1^2}{l}\left(1 - \frac{x}{l}\right) \quad . \quad 74.$$

Evidently S has its maximum positive value, for $x = 0$; its greatest negative value for $x = x_1$, and the value zero for $x = \dfrac{x_1}{1 + \dfrac{x_1}{l}}$.

In order to find the head of the moving load for that position which makes R a maximum, let $\dfrac{dR}{dx_1}$ be put equal to zero.

There is then found $x_1 = \dfrac{l}{2}$. Hence *the moving load covering half the span gives the maximum reaction R.*

Placing $x_1 = \frac{l}{2}$ in Eq. 72, the greatest value of R becomes $R_1 = w' l \div 4$.

In order to determine the greatest web stresses it is necessary to find the greatest shear at any point whose abscissa is x. This maximum shear at once results by placing the first derivative of S in respect to x_1, from Eq. 74, equal to zero. That operation gives:

$$1 - \frac{2x_1}{l}\left(1 - \frac{x}{l}\right) = 0 \therefore x_1 = \frac{l^2}{2(l-x)}.$$

By the introduction of this value of x_1 in Eq. 74, the greatest shear for any section located by x becomes:

$$max.\ S = w' l\left(\frac{1}{\left(1 - \frac{x}{l}\right)} - \frac{x}{l}\right) \quad . \quad . \quad . \quad . \quad 75.$$

It is clear that this value is a maximum for the reason that $\frac{d^2 S}{dx_1^2}$ is a negative quantity in which x_1 does not appear.

It is further evident that *max.* S is a positive, or upward shear, from the fact already observed that the greatest negative shear occurs at the head of the moving load. By making $x = x_1$ in Eq. 74, that greatest negative shear becomes:

$$S_1 = -w' \frac{x_1^2}{l}\left(1 - \frac{x_1}{l}\right) = -w' x_1 \frac{x_1}{l}\left(1 - \frac{x_1}{l}\right) \quad . \quad 76.$$

It will be necessary to apply *max.* S to a half of the span only, but Eq. 76 will be applied to the entire span, regarding the shear for a positive direction on one side of the centre as negative for the other. These two values of the shear will enable all the greatest web stresses to be determined.

Eqs. 75 and 76 can be used in practice most conveniently by the aid of the following table, which is self-explanatory. The table gives numerical values by which $w' l$ is multiplied in Eq. 75 and $-w' x_1$ in Eq. 76:

$\frac{x}{l}$ or $\frac{x_1}{l}$	$\left(\frac{1}{4\left(1-\frac{x}{l}\right)}-\frac{x}{l}\right)$	$\frac{x_1}{l}\left(1-\frac{x_1}{l}\right)$
.05	.213	.0475
.1	.178	.09
.15	.144	.1275
.2	.1125	.16
.25	.083	.1875
.3	.0571	.21
.35	.0346	.2275
.4	.0167	.24
.45	.0015	.2475
.5	.0000	.25

The value of x in Eq. 75 will never exceed $\frac{1}{2}l$. Again since the quantity $\frac{x_1}{l}\left(1-\frac{x_1}{l}\right)$ is symmetrical in reference to the centre of the span, x_1 in that expression need not exceed $\frac{1}{2}l$. The greatest value of $\frac{x}{l}$ or $\frac{x_1}{l}$ in the table, therefore, does not exceed .5.

Since $R' = 0$, the whole truss will be subjected to bending moments of the same sign; such bending moments, in fact, as will put the upper chord in compression and the lower one in tension.

The general value of the bending moment for that portion of the truss covered by the moving load is:

$$M = R\,x - (w + w' - t)\frac{x^2}{2},$$

$$\therefore M = w'\,x\left(1-\frac{x_1}{l}\right)\left(x_1-\frac{x}{2}\left(1+\frac{x_1}{l}\right)\right) \quad . \quad . \quad 77.$$

Since the shear S is zero for $x = \frac{l\,x_1}{l + x_1}$, that value of x in Eq. 77 will give the maximum value of M. This latter is:

$$M_1 = w'\,x_1^2\,\frac{l - x_1}{2\,(l + x_1)} = w'\,l^2\frac{x_1^2}{l^2}\,\frac{1-\frac{x_1}{l}}{2\left(1+\frac{x_1}{l}\right)} \quad . \quad . \quad 78.$$

Putting $\frac{dM}{dx_1} = 0$, there is found:

$$x_1 = \frac{l}{2}(-1 \pm \sqrt{5}) = +.618\,l.$$

The absolute maximum bending moment exists, therefore, when the moving load covers .618 of the span. That moment has for its value:

$$.0451\,w'l^2 \text{ (nearly)} \quad . \quad . \quad . \quad . \quad . \quad 79.$$

If $x_1 = .618\ l$ be put in the expression $x = \frac{l\,x_1}{l + x_1}$, there will result $x = .382\,l$, nearly.

Eq. 77 gives the general value of the bending moment for any position of the load. It may be advisable to design the chords of the stiffening-truss to resist that moment for the central portion $2\,(.5 - .382)\,l = .256\,l$, and with uniformly diminishing resistance over the end portions whose lengths are each $.382\,l$. By this arrangement there would be a little surplus of material at the middle of the truss.

The following table gives the numerical values of the quantity by which $w'l^2$ is to be multiplied in Eq. 78 in order to obtain

$\frac{x_1}{l}$	$\frac{x_1^2}{l^2}\,\frac{1 - \frac{x_1}{l}}{2\left(1 + \frac{x_1}{l}\right)}$
.05	.0013
.1	.0041
.15	.0083
.2	.0134
.25	.0188
.3	.0243
.35	0295
.4	.0344
.45	.0385
.5	.0417

the maximum bending moment for any point. A curve for practical use may readily be constructed from it; such a curve

would be the influence line for the greatest moments, *i. e.*, the values in the table are the greatest moments for a unit load w' and a unit length l of span.

If a single weight rests upon the bridge, the two general equations of equilibrium, obtained in precisely the same manner as heretofore, are:

$$w l + W - t l - R = 0 \quad . \quad . \quad . \quad . \quad 80.$$

$$(w - t)\frac{x_1^2}{2} - (w - t)\frac{(l - x_1)^2}{2} - R x_1 = 0 \quad . \quad 81.$$

These equations then give:

$$t = w + \frac{2 W x_1}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 82.$$

$$R = W\left(1 - \frac{2 x_1}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad 83.$$

If $x_1 = \frac{l}{2}$, $t = w + \frac{W}{l}$, and $R = 0$.

The general values of the shear S, and moment M, are the following:

$$S = R + (t - w)\, x,$$

$$\therefore S = W\left(1 - \frac{2 x_1}{l}\right) + \frac{2 W x_1 x}{l^2} \quad . \quad . \quad . \quad 84.$$

$$M = R x + (t - w)\frac{x^2}{2},$$

$$\therefore M = W\left(1 - \frac{2 x_1}{l}\right) x + \frac{W x_1 x^2}{l^2} \quad . \quad . \quad . \quad 85.$$

Eqs. 84 and 85 show, since x_1 must not be taken less than x, that if the maximum shear and bending moment are desired for any section, the weight W must be placed at that section, and x be made x_1 in those equations.

That section at which the bending moment will attain its absolute maximum value is found by putting $x = x_1$ in Eq. 85, then taking the first differential coefficient of M in respect to x_1, equating to zero and solving. There results:

$$x_1 = \tfrac{2}{3} l \pm \tfrac{1}{3} l = \tfrac{1}{3} l.$$

This value in Eq. 85, when $x = x_1$, gives:

$$M = \tfrac{4}{27} W l.$$

The equations for the continuous and single moving loads, used in combination, will give moments and shears for any character and position of loading whatever.

The general observations in regard to finding web and chord stresses, at the close of the last Article, apply equally well to this case.

As was to be anticipated, R and R' in the two preceding Articles are independent of the fixed load w.

Art. 13.—Approximate Character of the Preceding Investigations —Deflection of the Truss.

In the two preceding Articles it has been virtually assumed that the deflection of the cable, due to its lengthening under stress, is just sufficient to allow the truss to take the deflection due to the loads $T = p\,t$, w', and W in the different cases. This is not strictly true, but the error is not large.

It has been inferred from experiment that if the ends of the truss are anchored, the stiffening-truss will be subjected to a maximum moment approximately equal to that existing in an ordinary truss supported at each end, with the same span, and carrying load over its entire length of about one-eighth the intensity of the moving load on the suspension bridge.

This same case, treated analytically, as was seen, gave about one-seventh instead of one-eighth.

Approximate values of the deflection of the stiffening-truss can be found by the ordinary formulæ used for solid beams in the subject of resistance of materials, in the different cases, where t, R, R', and the moving load are known.

CHAPTER II

Art. 14.—Theory of the Stiffening-Truss with Centre Hinge.

THE theory of the stiffening-truss with a hinge at mid-span includes the condition that the bending moment about the centre of the hinge must always be zero, whatever may be the elastic deformation of the cables or suspenders. The fulfilment of this condition adds a corresponding equation of condition for the equilibrium of the truss, making three in all, which are sufficient to establish the two end reactions and the uniform pull s on the suspenders. Consequently the problem is statically determinate for this form of stiffening-truss. Although the stresses in the trusses, as well as in the suspenders, may be affected to some extent by the elastic deformation of those members, those effects are small and may generally be ignored; in any case, however, the main analysis presents only statical determination. In the following analysis all small effects due to elastic deformation of parts of the structure will be ignored.

The general scheme of treatment and the notation will be the same as employed in the preceding Articles.

The intensity of the uniform moving load is w, R and R' being the two reactions at the ends of the span A and B, Fig. 1, respectively. The uniform moving load will be taken as moving on the truss from A toward B. The co-ordinate x is measured from A toward C, the centre of the span, and x_1 will represent the length of train of uniform density w. The ends of the trusses at A and B will be taken as anchored down at those points.

Two cases may arise in one of which, Case 1, the uniform

moving load extends continuously from A toward C over the distance x_1; in the other, Case 2, the moving load is taken as extending from the centre of span C continuously toward A, the

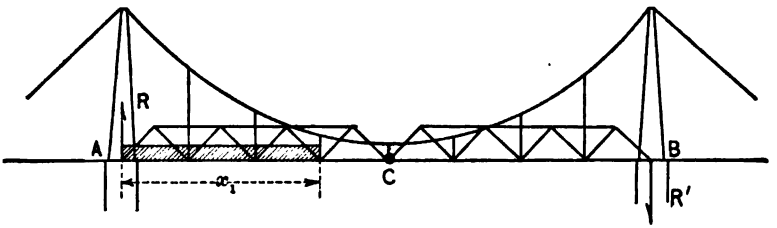

FIG. 1.

left-hand end of that uniform moving load reaching to a point distant x_1 from A.

CASE I.

Art. 15.—Moving Load Extending Continuously from A toward C.

Inasmuch as the pull s per linear unit (foot) exerted on the truss by the suspenders is upward, and as there is no moving load on $B\ C$, it is clear that the reaction R' at B must be downward in direction, and that an equal downward reaction must be exerted by the half truss $A\ C$ on $B\ C$ at C, the hinge point at the centre of the span. The load $w\,x_1$ on $A\ C$ may be supposed to be divided between A and C according to the law of the lever applied to the span $A\ C = \frac{l}{2}$. The reaction at C under that supposition is $\frac{w\,x_1^2}{l}$, and less than that at A. The total downward pull on the suspenders of the entire span will be $2\ \frac{w\,x_1^2}{l}$. This requires that there shall be a downward reaction or pull at B equal to one-half the total downward reaction at C, *i. e.*, $\frac{w\,x_1^2}{2\,l}$. Furthermore there will be carried directly to

the abutment A, without acting on the suspenders, the amount of load:

$$w x_1 \left(1 - \frac{x_1}{l}\right) - \frac{w x_1^2}{l} = w x_1 \left(1 - \frac{2 x_1}{l}\right).$$

The first member of this equation is evidently the reaction at A, supposing that $A C$ is an ordinary truss simply supported at A and C. When $x_1 = \frac{l}{2}$ in the above equation, the second member reduces to zero.

The forces acting on the entire truss $A B$ are then: the upward reaction R at A, the downward reaction R' at B, the upward pull $s l$ of the suspenders, and the moving load $w x_1$ acting downward; it is necessary for equilibrium that their sums shall be zero.

Hence:

$$s l + R - w x_1 - R' = 0 \quad . \quad . \quad . \quad . \quad 1.$$

Because the truss is hinged at the point C, the sum of the moments of the same forces about that point must be zero. Therefore:

$$\frac{s l^2}{8} + \frac{R l}{2} - w x_1 \left(\frac{l - x_1}{2}\right) - \frac{s l^2}{8} + \frac{R' l}{2} = 0 \quad . \quad 2.$$

Hence:

$$R = w x_1 \left(1 - \frac{x_1}{l}\right) - R' \quad . \quad . \quad . \quad . \quad 3.$$

By substituting this value of R in Eq. 1:

$$\frac{s l^2}{2} - R' l = \frac{w x_1^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 4.$$

If moments of the forces acting on the part $B C$ of the truss, Fig. 1, be taken about the centre of the hinge where the moment is zero:

$$-\frac{s l^2}{4} + R' l = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad 5.$$

By adding Eqs. 4 and 5:

$$R' = w x_1 \frac{x_1}{2 l} \quad . \quad . \quad . \quad . \quad . \quad . \quad 6.$$

This value of R' is the same as for a non-continuous beam of span l. By placing it in Eq. 3:

$$R = w x_1 \left(1 - \frac{3 x_1}{2 l}\right) \quad . \quad . \quad . \quad . \quad . \quad 7.$$

If the value of R' from Eq. 6 be placed in Eq. 5:

$$s = \frac{2 w x_1^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 8.$$

These values of R', R and s, in which x_1 must never exceed $\frac{1}{2} l$, will enable all moments and shears to be determined.

Art. 16.—Bending Moments in the Stiffening-Truss, Continuous, Load Moving from A toward C,

The expression for the bending moment at any point x (not greater than x_1) from the point of support A is:

$$M = R x + \frac{s x^2}{2} - \frac{w x^2}{2} = w \left\{ x_1 \left(1 - \frac{3 x_1}{2 l}\right) x + \left(\frac{2 x_1^2}{l^2} - 1\right) \frac{x}{2} \right\} \quad 9.$$

In order to find the position of loading for the maximum bending moment at any section distant x from the support A, the first derivative of M in respect to x_1 (x being considered constant) must be put equal to zero and then solved for x_1:

$$\frac{d M}{d x_1} = w \left\{ x - \frac{3 x_1 x}{l} + \frac{2 x_1 x^2}{l^2} \right\} = 0.$$

Hence:

$$x_1 = \frac{l}{3 - \frac{2 x}{l}} \quad . \quad . \quad . \quad . \quad . \quad . \quad 10.$$

By placing this value of x_1 in the third member of Eq. 9:

$$M_x = \frac{w l^2}{2} \frac{x}{l} \left(\frac{1}{3 - \frac{2 x}{l}} - \frac{x}{l} \right)^* \quad . \quad . \quad . \quad 11.$$

Eq. 11 will give the greatest possible bending moment in the half truss $A C$ at any point located by the co-ordinate x

* The use of this and other equations of this Chapter can be much expedited by employing the tables at the end of the Chapter.

which must never exceed $\frac{1}{2}\,l$. The corresponding length of loading from A will be given by x_1 in Eq. 10.

The greatest bending moment in the half truss $A\,C$ can be found by placing the first derivative of M_x in respect to x equal to zero and then solving that equation for x and inserting the latter value in Eq. 11, but there is a simpler procedure.

Let x_1 be considered constant in Eq. 9, then if $\frac{d\,M}{d\,x} = 0$, there will be located the section of greatest bending for any given length of loading x_1. Hence:

$$\frac{d\,M}{d\,x} = w\left\{x_1\left(1 - \frac{3\,x_1}{2\,l}\right) + \left(\frac{2\,x_1^2}{l^2} - 1\right)x\right\} = 0,$$

and,

$$x = x_1\,\frac{\frac{3\,x_1}{2\,l} - 1}{\frac{2\,x_1^2}{l^2} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad 12.$$

If this value of x be placed in Eq. 10:

$$\left(\frac{x_1}{l}\right)^3 - \frac{x_1}{l} + \tfrac{1}{3} = 0 \quad . \quad . \quad . \quad . \quad . \quad 13.$$

Eq. 13 is satisfied for $\frac{x_1}{l} = .395$ . . . . . 14.

Or, for this case the greatest bending moment in the span occurs when .395 of its length is covered with the moving load.

If $\frac{x_1}{l}$ is taken from Eq. 14 and placed in either Eq. 12 or Eq. 10:

$$\frac{x}{l} = .234 \quad . \quad . \quad . \quad . \quad . \quad . \quad 15.$$

Eq. 15 thus locates the section of greatest bending in the span. By placing $x = .234\,l$ in Eq. 11 that maximum moment becomes:

$$M_{max} = .018837\,w\,l^2 = .1507\,\frac{w\,l^2}{8} \quad . \quad . \quad . \quad 16.$$

The half truss $B\,C$ is simply supported at each end and carries the uniform upward load s per linear foot, giving the negative moment (measuring x from B):

$$M' = -\frac{s\,x}{2}\left(\frac{1}{2}\,l - x\right) = -w\frac{x_1^2\,x}{l^2}\left(\frac{1}{2}\,l - x\right) \quad . \quad 17.$$

This moment evidently has its greatest value, for $x_1 = \frac{1}{2}\,l$. Hence:

$$M'_{max} = -w\frac{l^2}{4}\frac{x}{l}\left(\frac{1}{2} - \frac{x}{l}\right). \quad . \quad . \quad . \quad 18.$$

In these expressions neither x_1 nor x can ever exceed $\frac{1}{2}\,l$.

When $x = \frac{1}{4}\,l$:

$$M'_{max} = -\frac{1}{8}\,\frac{w\,l^2}{8} = -\frac{w\,l^2}{64} \quad . \quad . \quad . \quad . \quad 19.$$

Art. 17.—Shears in the Stiffening-Truss, Continuous Load Moving from A toward C.

The shear at any section located by x (never exceeding x_1) is:

$$S = s\,x + R - w\,x = \frac{2\,w\,x_1^2\,x}{l^2} + w\,x_1\left(1 - \frac{3\,x_1}{2\,l}\right) - w\,x \quad . \quad 20.$$

$$\therefore S = w\,x\left(\frac{2\,x_1^2}{l^2} - 1\right) + w\,x_1\left(1 - \frac{3\,x_1}{2\,l}\right) \quad . \quad . \quad 21.$$

The greatest shear at any point or section x is determined by the same procedure employed to find the greatest moment:

$$\frac{d\,S}{d\,x_1} = \frac{4\,w\,x_1\,x}{l^2} + w - w\,\frac{3\,x_1}{l} = 0 \therefore x_1 = \frac{l^2}{3\,l - 4x} \quad . \quad 22.$$

By placing this value of x_1 in Eq. 21:

$$S_1 = w\,l\left(\frac{1}{2\left(3 - 4\frac{x}{l}\right)} - \frac{x}{l}\right) \quad . \quad . \quad . \quad . \quad 23.$$

This value of S_1 is the greatest shear in the half truss at

any section x. The form of Eq. 23 shows that the greatest of all the values of S_1 will be found by making $x = 0$. Hence:

$$S_{max} = \frac{wl}{6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 24.$$

By making $x = 0$ in Eq. 22, it appears that the length of moving load required to produce the maximum shear is:

$$x_1 = \frac{l}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 25.$$

This value of x_1 placed in Eq. 21 shows that the greatest shear in the half span is the end reaction R.

The preceding equations have been so written that an upward shear is positive and a downward shear negative. If S_1 be placed equal to zero, by using Eq. 23:

$$x = \frac{l}{4} \text{ or } \frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 26.$$

The second value indicates nothing useful, but the first value, $\frac{l}{4}$, shows that the shear is always upward over that quarter of the span nearest A, and downward or negative over that quarter adjacent to C. The greatest negative shears will obviously occur at the head of the moving load and they will be given by making $x = x_1$ in Eq. 21:

$$S_1 = w x_1 \frac{x_1}{l} \left(\frac{2 x_1}{l} - \frac{3}{2} \right) \quad . \quad . \quad . \quad . \quad . \quad 27.$$

For a maximum:

$$x_1 = \frac{l}{2} \text{ and } S_1 = -\frac{wl}{8}.$$

The preceding reasoning shows that *the maximum negative shears will be given by Eq. 27 in which x_1 must have values between $\frac{l}{4}$ and $\frac{l}{2}$ only; but that the maximum positive shears will be given by Eq. 23, in which x must have values between $x = 0$ and $x = \frac{l}{4}$ only.*

Since for $x_1 = \frac{l}{2}$, $R' = \frac{wl}{8}$, and $s = \frac{w}{2}$, the expression for

the shear in the half truss $B\,C$ will be, if x be measured from B:

$$S' = -w\left(\frac{l}{8} - \frac{x}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad 28.$$

By increasing x from o to $\frac{l}{2}$ in Eq. 28, there will result the negative shears in the right half of $B\,C$ and the positive shears in the left half of $B\,C$.

CASE II.

Art. 18.—Moving Load Extending Continuously from Centre of Span to Point Distant $(\frac{1}{2}\ l - x_1)$ from that Centre.

All notation remains the same as for Case I, but reference is made to Fig. 2.

The equations of condition for equilibrium, corresponding to Eqs. 1, 2 and 5 of Art. 15, are:

$$s\,l + R - w\left(\frac{l}{2} - x_1\right) - R' = o \quad . \quad . \quad . \quad 29.$$

$$\left.\begin{aligned} \frac{s\,l^2}{8} + \frac{R\,l}{2} - \frac{w}{2}\left(\frac{l}{2} - x_1\right)^2 &= o \\ -\frac{s\,l^2}{8} + \frac{R'\,l}{2} &= o \end{aligned}\right\} \quad . \quad . \quad . \quad 30.$$

The two Eqs. 30 give:

$$R = \frac{w}{l}\left(\frac{l}{2} - x_1\right)^2 - R' \quad . \quad . \quad . \quad . \quad 31.$$

The substitutions of R from Eq. 31 and $s\,l$ from the second of Eqs. 30 in Eq. 29 give:

$$R' = \frac{w}{2}\left(\frac{l}{2} - x_1\right)\left(\frac{x^1}{l} + \frac{1}{2}\right) = \frac{w\,l}{2}\left(\frac{1}{4} - \frac{x_1^2}{l^2}\right) \quad . \quad 32.$$

By placing this value of R' in Eq. 31:

$$R = \frac{w\,l}{2}\left(\frac{1}{2} - \frac{3\,x_1}{l}\right)\left(\frac{1}{2} - \frac{x_1}{l}\right) \quad . \quad . \quad . \quad 33.$$

The same value of R' in the second of Eqs. 30:

$$s = 2w\left(\frac{1}{4} - \frac{x_1^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad . \quad 34.$$

The reactions R and R' will evidently have their greatest values when $x_1 = 0$:

$$R_{max} = R'_{max} = \frac{wl}{8} \quad . \quad . \quad . \quad . \quad . \quad 35.$$

The equality between these reactions is numerical only, as R' acts downward and R upward.

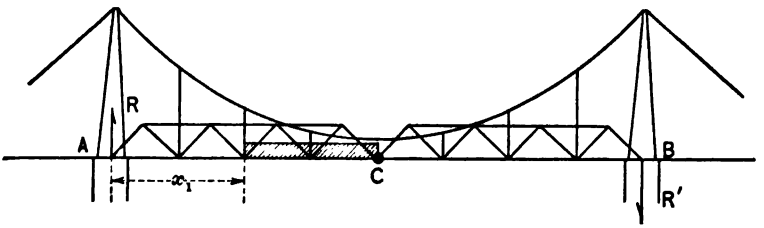

FIG. 2.

Art. 19.—Bending Moments, Continuous Load Moving from C toward A.

The greatest value of two bending moments must be determined, *i. e.*, one at the extremity of the moving load at the distance x_1 from A, Fig. 2, and the other at any point of $A\,C$ with that half span entirely covered with the moving load.

The moment at any point x between the end A and x_1 is:

$$M = Rx + \frac{1}{2}s\,x^2 = wxl\left(\frac{1}{2} - \frac{x_1}{l}\right)\left\{\frac{1}{2}\left(\frac{1}{2} - \frac{3\,x_1}{l}\right) + \frac{x}{l}\left(\frac{1}{2} + \frac{x_1}{l}\right)\right\} \quad . \quad 36.$$

The form of this equation shows its value to be a maximum for $x = x_1$:

$$M_1 = w\,l^2\frac{x_1}{l}\left(\frac{1}{2} - \frac{x}{l}\right)^3 \quad . \quad . \quad . \quad . \quad . \quad 37.$$

In order to find a maximum:

$$\frac{d\,M_1}{d\,x_1} = w\,l\left\{\left(\frac{1}{2} - \frac{x_1}{l}\right)^3 - 3\frac{x_1}{l}\left(\frac{1}{2} - \frac{x_1}{l}\right)^2\right\} = 0 \quad . \quad 38.$$

Hence:

$$x_1 = \frac{l}{8}.$$

By placing this value of x_1 in Eq. 37:

$$M_{1\,max} = \frac{w\,l^2}{8}\,\frac{27}{512} = .0527\,\frac{w\,l^2}{8} \quad . \quad . \quad . \quad . \quad 39.$$

Eq. 39 gives the greatest possible bending moment in the partially loaded half span.

If the moving load covers the whole of the half span $A\,C$, the moment at any point x will be, since then $s = \frac{w}{2}$ and $R = \frac{w\,l}{8}$;

$$M^1 = R\,x - (w - s)\,\frac{x}{2} = w\,l^2\,\frac{x}{l}\left(\frac{1}{8} - \frac{1}{4}\,\frac{x}{l}\right) \quad . \quad . \quad 40.$$

This is a maximum for $x = \frac{l}{4}$:

$$M^1{}_{max} = .125\,\frac{w\,l^2}{8} = \frac{w\,l^2}{64} \quad . \quad . \quad . \quad . \quad 41.$$

Thus both the general value of the bending moment and its greatest value for the fully loaded half span are given by Eqs. 40 and 41.

The bending moments in the unloaded half span are given by Eqs. 18 and 19 of the preceding case.

Art. 20.—Shears in the Stiffening-Truss, Continuous Load Moving from C toward A.

The general expression for the upward (positive) shear in the unloaded portion x_1 of the left half $A\,C$ of the span is:

$$S = R + s\,x = w\,l\,\left(\frac{1}{2} - \frac{x}{l}\right)\left(\frac{1}{4} - \frac{3\,x_1}{2\,l} + \frac{2\,x\,x_1}{l^2} + \frac{x}{l}\right) \quad . 42.$$

For the greatest value of the shear S, $x = x_1$:

$$S_m = w\,l\,\left\{\frac{1}{2}\left(\frac{1}{2} - \frac{x_1}{l}\right)^2 + \frac{2\,x_1^2}{l^2}\left(\frac{1}{2} - \frac{x_1}{l}\right)\right\} = w\,l\,X_1 \quad . 43.$$

There can be no negative shear in that half of the span carrying the moving load.

The shears in the right half of the span with the left half $A\,C$ fully loaded have been given in the preceding case by Eq. 28.

These expressions complete all the equations required for the treatment of the three-hinged stiffening-truss subjected to a continuous uniform moving load.

Art. 21.—The Three-Hinged Stiffening-Truss Subjected to a Single Moving Load, W.

This analysis, including the notation, is practically like that employed for a uniform moving load in the preceding pages. The single weight W at the distance x_1 from A, Fig. 3, is the entire moving load. It is only necessary to consider W placed in one half span, as the results would simply be duplicated by placing it in the other.

The three equations of condition for statical equilibrium are the following:

$$s\,l + R - W - R' = 0 \quad . \quad . \quad . \quad . \quad 44.$$

$$\left.\begin{aligned} \frac{s\,l^2}{8} + \frac{R\,l}{2} - W\left(\frac{l}{2} - x_1\right) &= 0 \\ -\frac{s\,l^2}{8} + \frac{R'\,l}{2} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad 45.$$

Adding the two equations 45:

$$R + R' - W\left(1 - \frac{2\,x_1}{l}\right) = 0 \quad . \quad . \quad . \quad 46.$$

Adding Eqs. 44 and 46:

$$s\,l + 2\,R - 2\,W\left(1 - \frac{x_1}{l}\right) = 0 \quad . \quad . \quad . \quad 47.$$

Combining the first of Eqs. 45 and Eq. 47:

$$R = W\left(1 - \frac{3\,x_1}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad 48.$$

Eq. 46 then gives:

$$-R' = -W\frac{x_1}{l} \quad . \quad . \quad . \quad . \quad . \quad 49.$$

The negative sign is used for R' to indicate that it acts downward. The value of R' is the same as for a simply supported beam of length l.

The second of Eqs. 45 now gives:

$$s = \frac{4 W x_1}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 50.$$

In all these equations obviously x_1 must never exceed $\frac{l}{2}$.

The bending moments and shears in the stiffening-trusses follow at once from the preceding equations.

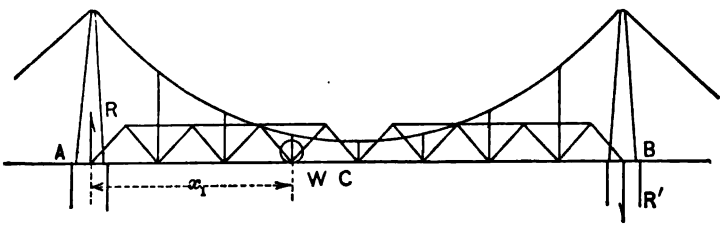

FIG. 3.

Art. 22.—Bending Moments in the Stiffening-Truss, for Single Moving Load, W.

Three values of bending moments must now be written, one for the part x_1 of the span, one for the part $\frac{l}{2} - x_1$, and one for the unloaded half span $B\,C$. The moment for the part x_1 is:

$$M = R\,x + \frac{s\,x^2}{2} = W\,x\left(1 - \frac{3\,x_1}{l} + \frac{2\,x\,x_1}{l^2}\right) \quad . \quad 51.$$

The moment for the part $\left(\frac{l}{2} - x_1\right)$ is:

$$M' = R\,x + \frac{s\,x^2}{2} - W(x - x_1) = W\,x_1\left(1 - \frac{3\,x}{l} + \frac{2\,x^2}{l^2}\right) \quad . \quad 52.$$

The forms of Eqs. 51 and 52 show that Eq. 51 takes its greatest value for the greatest value of x, *i. e.*, x_1, and Eq. 52

for the smallest value of x, *i. e.*, also x_1. This common greatest bending moment is then:

$$M_{max} = W x_1 \left(1 - \frac{3 x_1}{l} + \frac{2 x_1^2}{l^2}\right) \quad . \quad . \quad . \quad 53.$$

The bending moment for the unloaded half span is, measuring x from B:

$$M = -\frac{s x^2}{2} + R' x = W \frac{x_1 x}{l} \left(1 - \frac{2 x}{l}\right) \quad . \quad . \quad 54.$$

This expression has its maximum value for $x = \frac{l}{4}$, the centre of the unloaded half span. That value of x gives:

$$M_{max} = \frac{W x_1}{8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 55.$$

This completes all required values of bending moments.

Art. 23.—Shears in the Stiffening-Trusses, for Single Moving Load, W.

Three values for shears are to be written, as for the bending moments, one for the part x_1 of the span, one for the part $\frac{1}{2} l - x_1$, and the third for the unloaded half of span. The first of the three values in which x must never exceed x_1 is:

$$S = R + s x = W \left(1 - \frac{3 x_1}{l} + \frac{4 x_1 x}{l^2}\right) \quad . \quad . \quad 56.$$

The second value, for the part $\frac{1}{2} l - x_1$ of the span, is:

$$S^1 = R + s x - W = -W \frac{x_1}{l} \left(3 - 4 \frac{x}{l}\right) \quad . \quad 57.$$

Eq. 56 has its greatest value for $x = x_1$:

$$S_{max} = W \left(1 - \frac{3 x_1}{l} + \frac{4 x_1^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad 58.$$

Eq. 57 has its greatest negative value for $x = x_1$:

$$S'_{max} = -W \frac{x_1}{l} \left(3 - 4 \frac{x}{l}\right) \quad . \quad . \quad . \quad . \quad 59.$$

The least negative value will be found for $x = \frac{l}{2}$;

$$S'_{min} = -W\frac{x_1}{l} \quad . \quad . \quad . \quad . \quad . \quad 60.$$

The shear in the unloaded half span, measuring x from B, is:

$$S_1 = -R' + s\,x = -W\frac{x_1}{l}\left(1 - \frac{4x}{l}\right) \quad . \quad . \quad 61.$$

Eq. 61 has a negative maximum for $x = 0$:

$$S_1 = -W\frac{x_1}{l} = -R' \quad . \quad . \quad . \quad . \quad 62.$$

Also:

$$S_1 = 0 \text{ for } x = \frac{l}{4} \quad . \quad . \quad . \quad . \quad . \quad 63.$$

Finally, S_1 has a positive maximum for $x = \frac{l}{2}$:

$$S_1 = W\frac{x_1}{l} = R^1 \quad . \quad . \quad . \quad . \quad . \quad 64.$$

These expressions complete all values of the shears for the entire span.

TABLES.

The following tabular values aid greatly the convenient use of the equations of the preceding Articles. Curves showing continuous values graphically may readily be constructed from them:

$\frac{x_1}{l}$	$\left(1-\frac{3}{2}\frac{x_1}{l}\right)$	$\frac{x_1}{2l}$	$\frac{2x_1^2}{l^2}$	$\frac{x}{l}\left(\frac{1}{2}-\frac{x}{l}\right)$	$\frac{1}{2\left(3-4\frac{x}{l}\right)}-\frac{x}{l}$	$\frac{x_1}{l}\left(\frac{2x_1}{l}-\frac{3}{2}\right)$
.05	.925	.025	.005	.0225	.1286	−.07
.1	.85	.05	.02	04	.0923	−.13
.15	.775	.075	.045	.0525	.0583	−.18
.2	.7	.1	.08	.06	.0273	−.22
.25	.625	.125	.125	.0625	.0	−.25
.3	.55	.15	.18	.06	−.0222	−.27
.35	.475	.175	.245	.0525	−.0375	−.28
.4	.4	.2	.32	.04	−.0429	−.28
.45	.325	.225	.405	.0225	−.0333	−.27
.5	.25	.25	.5	.0	0	−.25

$\frac{x_1}{l}$	$\left(\frac{1}{4}-\frac{x_1^2}{l^2}\right)$	$\left(\frac{1}{2}-\frac{x_1}{l}\right)\left(\frac{1}{2}-\frac{3x_1}{l}\right)$	$\frac{x_1}{l}\left(\frac{1}{2}-\frac{x_1}{l}\right)^3$	X_1 Eq. 43	$\left(1-\frac{3x_1}{l}\right)$	$\left(1-\frac{3x}{l}+\frac{2x^2}{l^2}\right)$
.05	.2475	.1575	.004556	.1035	.85	855
.1	.24	.08	.0064	.088	.7	.72
.15	.2275	.0175	.006431	.077	.55	595
.2	.21	−.03	.0054	.069	.4	.48
.25	.1875	−.0625	.003906	.0625	.25	.375
.3	.16	−.08	.0024	.056	.1	.28
.35	.1275	−.0825	.001181	.048	−.05	.195
.4	.09	−.07	.0004	.037	−.2	.12
.45	.0475	−.0425	.0000563	.0215	−.35	.055
.5	.0	−.0	.0	.0	−.5	.0

$\frac{x_1}{l}$	$\frac{x}{l}\left(\frac{1}{3-\frac{2x}{l}}-\frac{x}{l}\right)$	$\left(1-\frac{3x_1}{l}+\frac{4x_1^2}{l^2}\right)$	$\frac{x_1}{l}\left(3-4\frac{x_1}{l}\right)$	$\frac{x_1}{l}\left(1-\frac{3x_1}{l}+\frac{2x_1^2}{l^2}\right)$	$\frac{x}{l}\left(\frac{1}{8}-\frac{1}{4}\frac{x}{l}\right)$
.05	.01474	.86	.14	.04275	.005625
.1	.025714	.74	.26	.072	.01
.15	.033056	.64	.36	.08925	.013125
.2	.036964	.56	.44	.096	.015
.25	.0375	.5	.5	.09375	.015625
.3	.035	.46	.54	.084	.015
.35	.02968	.44	.56	.06825	.031125
.4	.02182	.44	.56	.048	.01
.45	.01179	.46	.54	.02475	.005625
.5	.0	.5	.5	.0	.0

CHAPTER III

Art. 24.—The Problem of the Stiffened Suspension Bridge in General.

The complete problem of the stiffened suspension bridge involves the treatment of the stiffening-trusses as statically indeterminate. Equations of condition depending upon the elastic properties of the material are therefore required, and they will be formulated in this and succeeding Chapters. Prior to doing so, however, it is best to set forth a thorough treatment of a perfectly flexible cable or frame loaded vertically.

Art. 25.—General Consideration of the Equilibrium Polygon for a Perfectly Flexible Cable or Frame Sustaining Vertical Loads.

If a weightless cable or frame in tension carrying any system of loads, W_1, W_2, W_3, etc., be represented in Fig. 1, its relation to the load or force polygon shown in Fig. 2 is at once apparent. The rays of the latter represent the directions of the sides of the former, and the stresses in those sides. The funicular polygon of Fig. 1 is constructed from the force diagram of Fig. 2 by the usual methods of graphical statics. Both figures remain completely unchanged, whether a frame in tension or compression be considered, except in the latter case Fig. 1 would be reversed so as to be convex upward instead of concave. This shows that the general treatment of flexible cables and arch ribs is identically the same, but it must be observed that the equilibrium of the loaded flexible cable is stable, while that of the frame in compression with frictionless hinged joints is unstable.

In Fig. 1, at the points of support A and B the vertical components of the resultant stresses T_1 and T_6 are V^a and V^b. As

the loading is wholly vertical no external horizontal force is applied to the cable or frame except at the two points of support A and B. Hence the two horizontal forces H must be equal

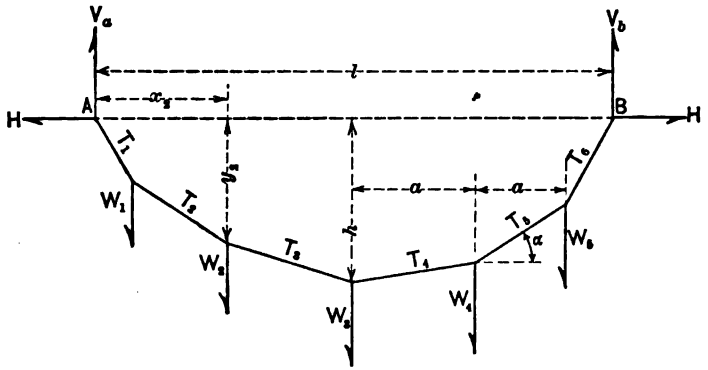

FIG. 1.

to each other but opposite in direction. For the same reason the horizontal component of stress in each member of the cable or frame must have the same value H.

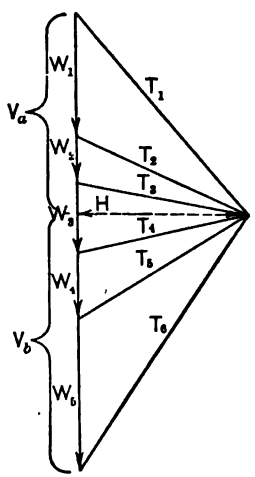

FIG. 2.

The vertical component of the stress in any side or member T_m of Fig. 1 is:

$$V_m = V_a - W_1 - W_2 - \ldots - W_{m-1} \ldots 1.$$

The angle α_m made by the same member with a horizontal line is given by:

$$tan\, \alpha_m = \frac{V_m}{H} \quad . \quad . \quad . \quad 2.$$

The resultant stress in the member is:

$$T_m = \sqrt{V_m^2 + H^2} = H\, sec\, \alpha_m \quad . \quad 3.$$

As H is constant the resultant stress is thus seen to be proportional to the secant of the inclination to the horizontal.

If x_m and y_m are horizontal and vertical co-ordinates, respectively, measured from A as an origin to the joints of the polygon, Fig. 1, there will result from similarity of triangles:

$$\frac{y_m - y_{m-1}}{x_m - x_{m-1}} = \frac{V_m}{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 4.$$

Or, there may be written from Eq. 4:

$$\Delta y_m = \frac{V_m}{H} \Delta x = \frac{V_a - W_1 - W_2 - \ldots - W_{m-1}}{H} \Delta x \quad . \quad 5.$$

A similar expression may obviously be written for any other member.

If it be assumed that the loads W_1, W_2, etc., are at a small and uniform horizontal distance apart, which is permissible for the purpose of analysis, so that the distance a (or Δx) is constant, the variation of the third member of Eq. 5 will be:

$$\Delta \left(\frac{V_a - W_1 - W_2 - \ldots - W_{m-1}}{H} \right) \Delta x = - \frac{W_{m-1}}{H} \Delta x \quad . \quad 6.$$

Taking the variation, therefore, of the first member of Eq. 5, and writing a for Δx:

$$\Delta^2 y_m = - \frac{W_{m-1}}{H} a \quad . \quad . \quad . \quad . \quad . \quad . \quad 7.$$

By writing a value of $tan\, \alpha_{m-1}$, with the aid of Eq. 2, the difference of consecutive tangents is found to be:

$$tan\, \alpha_m - tan\, \alpha_{m-1} = - \frac{W_{m-1}}{H} \quad . \quad . \quad . \quad . \quad 8.$$

It is obvious that the vertical reaction V_a may be obtained by the simple principle of the lever by taking moments about the other point of support, B, as for an ordinary non-continuous beam, as follows:

$$V_a = \frac{W_1 (l - x_1) + W_2 (l - x_2) + \text{etc.}}{l} = \frac{\Sigma W (l - x)}{l} \quad . \quad 9.$$

Again, the two points of support being in the same horizontal plane, and f being the height of those points above the lowest joint or angle of the frame at the distance x_a from A, the value of the horizontal force H may be written:

$$H = \frac{V_a x_a - W_1 (x_a - x_1) - W_2 (x_a - x_2) - \text{etc.}}{f} \quad . \quad 10.$$

Another expression for H may be given if the lengths of the

sides of the polygon are known. Let e_1, e_2, e_3, etc., be these lengths, the cosines of whose inclinations to the horizontal are:

$$\cos \alpha_a = \frac{H}{\sqrt{V_a^2 + H^2}}; \quad \cos \alpha_1 = \frac{H}{\sqrt{(V_a - W_1)^2 + H^2}};$$

$$\cos \alpha_2 = \frac{H}{\sqrt{(V_a - W_1 - W_2)^2 + H^2}}; \text{ etc., etc.}$$

The length of span l is the sum of the lengths e multiplied by $\cos \alpha$. Hence:

$$H = \frac{l}{\frac{e_1}{\sqrt{V_a^2 + H^2}} + \frac{e_2}{\sqrt{(V_a - W_1)^2 + H^2}} + \frac{e}{\sqrt{(V_a - W_1 - W_2)^2 + H^2}} + \text{etc.}} \quad \ldots \quad 11.$$

With a curved cable or arched rib, Eq. 11 would be approximate only, although the approximation might be made close.

These comprise the principal general equations forming the basis of analytical treatment of perfectly flexible cables for tension and for hinged frames in compression, the former being convex downward and the latter convex upward.

Art. 26.—Flexible Cables and Hinged Frames for Continuous Loading.

The usual engineering structures, such as suspension bridges and arched ribs, generally carry continuous loads distributed according to some simple law. The most common case is a uniform distribution of loading, although concentrated loads are sometimes employed and the catenary must be used for some conditions. It is only necessary to adapt the preceding general equations to the special conditions of each case in order to reach the desired solution of any particular problem.

If the loading is distributed according to any known law, and if w represents the intensity (*i.e.*, amount per horizontal

foot in American engineering practice) of that loading, the sum of $W_1 + W_2 + W_3 +$ etc. is to be replaced by $\int w\, dx$. Hence Eq. 5 of Art. 25 becomes:

$$dy = \frac{V_a - \int w\, dx}{H} dx \quad . \quad . \quad . \quad . \quad . \quad 12.$$

By differentiating once:

$$H \frac{d^2 y}{dx^2} = -w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 13.$$

It is shown in differential calculus that $\frac{d^2 y}{dx^2} = -\frac{1}{r} sec^3 \alpha$, in which r is the radius of curvature. Consequently:

$$H = \frac{w r}{sec^3 \alpha} = w r \cos^3 \alpha \quad . \quad . \quad . \quad . \quad . \quad 13a.$$

The substitution of this value of H in Eq. 3 of Art. 25 gives the general value of the cable or frame stress at any point:

$$T = w r \cos^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad 14.$$

At the vertex of the curve where its radius is r_0, and $\alpha = 0$, and where the vertical load per linear foot is w_0:

$$T_0 = H = w_0 r_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 15.$$

Art. 27.—The Parabolic Cable or Frame.

If the intensity of loading w is constant, and if Eq. 13 be integrated twice, taking the origin of co-ordinates at the vertex

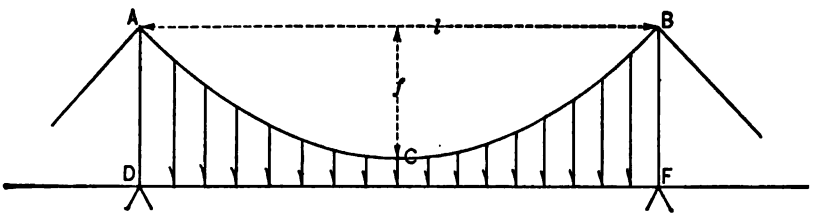

FIG. 3.

of the curve so that both $\frac{dy}{dx} = 0$ and $y = 0$ for $x = 0$, there will result:

$$y = \frac{w x^2}{2 H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 16.$$

This is the equation of the ordinary parabola, and it shows that if a perfectly flexible cable carries a load uniform over a straight line, such as the span, the curve of the cable will be a parabola with its axis parallel to the direction of loading.

If the two points of support are at the same elevation $y = f$, the versine or sag of the curve, and H takes the value:

$$H = \frac{w l^2}{8 f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 17.$$

Hence from Eq. 16:

$$y = 4 f \frac{x^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 18.$$

The greatest stress in the cable or frame will be at the points of support where the inclination α is the greatest and its value will be:

$$T_a = \sqrt{H^2 + (\tfrac{1}{2} w l)^2} = \frac{w l^2}{8 f} \sqrt{1 + \left(\frac{4 f}{l}\right)^2} \quad . \quad . \quad 19.$$

The radical in the third member of Eq. 19 is evidently the secant of inclination of the cable to the horizontal at the point of support.

Art. 28.—The Catenary.

A catenary is the curve assumed by a cable of uniform sectional area loaded by its own weight only, or carrying a load of uniform intensity along its length.

If g is the uniform load per linear unit (foot) along the curve, the load w per horizontal foot will be:

$$w = g \sec \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 20.$$

And $\sec \alpha = \sqrt{1 + \left(\frac{d y}{d x}\right)^2}$. Hence by Eqs. 13 and 20, neglecting the negative sign in connection with the radical:

$$H \frac{d^2 y}{d x^2} = g \left(1 + \left(\frac{d y}{d x}\right)^2\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad 21.$$

This differential equation may readily be integrated by first placing $p = \frac{dy}{dx}$ and giving Eq. 21 the form:

$$\frac{dp}{(1+p^2)^{\frac{1}{2}}} = \frac{g}{H}dx \quad . \quad . \quad . \quad . \quad . \quad 22.$$

The integration of Eq. 22 gives, as the origin of x and y is at the vertex of the curve:

$$\log.\ (p + (1+p^2)^{\frac{1}{2}}) = \frac{g}{H}x + (C = 0) \quad . \quad . \quad 23.$$

The constant of integration $C = 0$ because when $x = 0$, $p = 0$. Hence by writing $\frac{dy}{dx}$ for p and representing the base of the natural system of logarithms by e, also placing $c = \frac{g}{H}$:

$$\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = e^{cx} \quad . \quad . \quad . \quad . \quad 24.$$

Observing that $(dx^2 + dy^2)^{\frac{1}{2}} = ds$, the differential of the curve length, Eq. 24 may take the form:

$$ds = e^{cx}\,dx - dy.$$

Hence, integrating, the length of the curve from the vertex is:

$$s = \frac{1}{c}e^{cx} - y - \frac{1}{c} \quad . \quad . \quad . \quad . \quad . \quad 25.$$

As the origin of co-ordinates is at the vertex of the curve y and s are zero for $x = 0$ and the constant of integration is $-\frac{1}{c}$.

Again, transposing $\frac{dy}{dx}$ in Eq. 24 to the right-hand member and squaring both sides, there will result:

$$\frac{dy}{dx} = \tfrac{1}{2}e^{cx} - \tfrac{1}{2}e^{-cx} \quad . \quad . \quad . \quad . \quad . \quad 26.$$

Hence, as the origin of x and y is at the vertex of the curve:

$$y = \frac{1}{2c}\left(e^{cx} + e^{-cx} - 2\right) \quad . \quad . \quad . \quad . \quad 27.$$

Eq. 27 is the equation of the catenary, C, Fig. 4, being the vertex of the curve and the origin of co-ordinates. The length

of any part of the curve, as between C and D, is found conveniently by substituting y from Eq. 27 in Eq. 25:

$$s = \frac{1}{2c}\left(e^{cx} - e^{-cx}\right) \quad . \quad . \quad . \quad . \quad . \quad 28.$$

By comparing s^2 and y^2 by the aid of Eqs. 27 and 28 it may readily be seen, also, that:

$$s = \sqrt{y^2 + \frac{2y}{c}} \quad . \quad . \quad . \quad . \quad . \quad 29.$$

These are all the principal equations necessary for the solution of catenary problems.

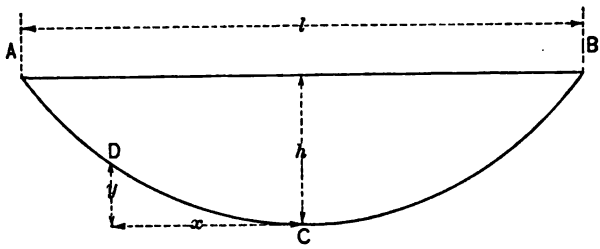

FIG. 4.

It is sometimes desirable to express x in terms of s and c. This may be done in the following manner:

$$tan\,\alpha = \frac{\sqrt{1 - \frac{dx^2}{ds^2}}}{\frac{dx}{ds}} = \frac{gs}{H} = cs \quad . \quad . \quad . \quad 30.$$

$$\therefore 1 - \frac{dx^2}{ds^2} = c^2 s^2 \frac{dx^2}{ds^2} \therefore \frac{dx}{ds} = \frac{1}{\sqrt{1 + c^2 s^2}} \quad . \quad . \quad 31.$$

Hence:

$$c\,dx = \frac{d(cs)}{\sqrt{1 + c^2 s^2}}; \text{ and: } x = \frac{1}{c} \log.\,(cs + \sqrt{1 + c^2 s^2}) \quad . \quad 32.$$

The trigonometric quantities giving the inclination of the curve to a horizontal line at any point are:

$$tan\,\alpha = \frac{dy}{dx} = \tfrac{1}{2} e^{cx} - \tfrac{1}{2} e^{-cx} \quad . \quad . \quad . \quad . \quad 33.$$

$$\sec \alpha = \frac{ds}{dx} = \tfrac{1}{2} e^{cx} + \tfrac{1}{2} e^{-cx} \quad . \quad . \quad . \quad . \quad 34.$$

$$\sin \alpha = \frac{dy}{ds} = \frac{c\sqrt{y^2 + \frac{2y}{c}}}{cy + 1} \quad . \quad . \quad . \quad . \quad 35.$$

Eq. 35 is obtained by dividing Eq. 33 by Eq. 34 and substituting values from Eqs. 27 and 29.

The preceding equations are readily applicable for the solution of all problems when the quantity $c = \frac{g}{H}$ is known. This quantity is the parameter of the curve.

In consequence of the exponential character of most of the equations in which c enters, its value must be found by trial except in the case where the length s of the curve and the ordinate y are known. Eq. 29 will give:

$$c = \frac{2y}{S^2 - y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 36.$$

If the two points of support A and B in Fig. 4 are at the same elevation, while l is the span and f the sag or dip of the cable; and finally if S is the known total length of the cable $A\,C\,B$, the value of c becomes:

$$c = \frac{2f}{\frac{1}{4}S^2 - f^2} = \frac{8f}{S^2 - 4f^2} \quad . \quad . \quad . \quad . \quad 37.$$

If f is not known, Eq. 32 gives:

$$e^{\frac{1}{2}cl} = \tfrac{1}{2} c s + \sqrt{1 + \tfrac{1}{4} c^2 S^2} \quad . \quad . \quad . \quad . \quad . \quad 38.$$

In this case c must be found by trial by the aid of Eq. 38. When a value of c which numerically satisfies Eq. 38 has been found or when it has been found from Eq. 37, the horizontal tension H at the vertex C of the curve at once results:

$$H = \frac{g}{c}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 39.$$

g being the known weight per linear foot of the cable.

If s is not known while f and l are given, Eq. 27 will take the form:

$$f = \frac{1}{2c}\left(e^{\frac{1}{2}cl} + e^{-\frac{1}{2}cl} - 2\right) \quad . \quad . \quad . \quad . \quad 40.$$

The parameter c is the only unknown quantity, and it can be found at once by trial.

If the elevations of the two points of support A and B, Fig. 5, are not the same the determinations of the two abscissæ

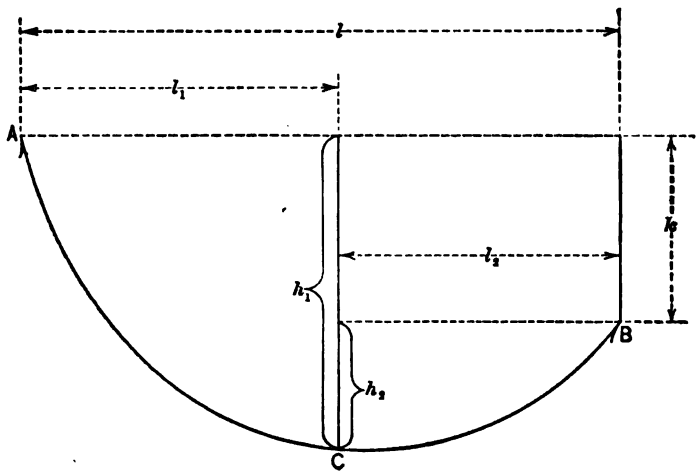

FIG. 5.

l_1 and l_2 of the vertex C of the cable, and of the parameter c, are not so simple. Let s_1 be the length of the curve $A\,C$ and s_2 the length of $B\,C$, while the total length $S = s_1 + s_2$ is supposed to be known. Then by Eq. 28:

$$S = \frac{1}{2c}\left(e^{cl_1} - e^{-cl_1} + e^{cl_2} - e^{-cl_2}\right) \quad . \quad . \quad . \quad 41.$$

Also by the aid of Eq. 27:

$$f_1 - f_2 = k = \frac{1}{2c}\left(e^{cl_1} - e^{cl_2} + e^{-cl_1} - e^{-cl_2}\right) \quad . \quad 42.$$

The two equations 41 and 42 contain but two unknown quantities c and l_1, when S and k are given, as $l_2 = l - l_1$, l being the length of span $A\,B$. Hence the values of c and l_1 may be found from them by trial. The abscissæ l_1 and l_2 thus computed or found will locate the vertex C. The ordinates f_1 and f_2 of the vertex may then at once be computed by Eq. 27.

When S is not known but f_1 and f_2 are given, Eq. 27 yields the two equations of condition:

$$f_1 = \frac{1}{2c}\left(e^{ch_1} + e^{-ch_1} - 2\right) \quad . \quad . \quad . \quad . \quad 43.$$

$$f_2 = \frac{1}{2c}\left(e^{ch_2} + e^{-ch_2} - 2\right) \quad . \quad . \quad . \quad . \quad 44.$$

As $l_2 = l - l_1$ there are two unknown quantities only in Eqs. 43 and 44, *i. e.*, c and l_1, and they can at once be found by trial. The length of the curve can then be computed by either Eq. 28 or 29.

In seeking a trial value of $c = \frac{g}{H}$, H can be taken in the first trial as if it belonged to a parabolic cable, and then subjected to a suitable correction.

After c has been determined, the horizontal component, H, of the cable stress is at once known:

$$H = \frac{g}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 45.$$

The cable stress at any point whose abscissa from the vertex is x at once results by the aid of Eq. 34:

$$T = H \sec \alpha = \tfrac{1}{2} H \left(e^{cx} + e^{-cx}\right) \quad . \quad . \quad . \quad 46.$$

The greatest value of T exists at the points of support where $x = \frac{1}{2} l$:

$$max.\ T = \tfrac{1}{2} H \left(e^{\frac{1}{2}cl} + e^{-\frac{1}{2}cl}\right) \quad . \quad . \quad . \quad . \quad 47.$$

At any point $H \tan \alpha = H\frac{dy}{dx} = gs$ is the vertical component of the cable stress, *i. e.*, the weight of the cable between that point and the vertex. Therefore by the aid of Eq. 33:

$$H \tan \alpha = g s = \tfrac{1}{2} H \left(e^{cx} - e^{-cx}\right) \quad . \quad . \quad . \quad 48.$$

In all the preceding equations, the quantity e, the base of the system of natural logarithms, has the value 2.71828, and its common logarithm is 0.4342945.

Art. 29.—Stresses in the Cable Caused by the Weight of the Structure and by a Load Uniform over the Span. Horizontal Component H of Cable Stress.

As the moving load will be assumed to be uniform over the entire span in this Article, the results will be equally applicable to unstiffened and stiffened suspension bridges for that condition of moving load.

Let the total own weight or dead load of the entire structure, including that of the cable and suspension rods, be supposed uniformly distributed over the span, which is sufficiently accurate for all practical purposes. It has been shown in Art. 27 that the curve of the cable will then be a parabola with the vertex at the lowest point. If g_0 is the weight of the chain or cable per unit of length at its lowest point or vertex, and if any normal sectional area of the cable is proportional to its total stress, and, finally, if its inclination to a horizontal line at any point is α, the weight g_x per linear foot of span of the chain or cable will be:

$$g_x = g_0 \sec \alpha \qquad \sec \alpha = g_0 \sec^2 \alpha.$$

By Eq. 18 of Art. 27:

$$\sec^2 \alpha = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{64 f^2 x^2}{l^4} \quad . \quad . \quad . \quad 49.$$

Hence:

$$g_x = g_0 \left(1 + \frac{64 f^2 x^2}{l^4}\right) \quad . \quad . \quad . \quad . \quad . \quad 50.$$

If g^1 be the weight of the suspenders per square unit of vertical plane of the cable, so that $g^1 y$ is the weight per linear horizontal foot of that part of the suspenders above the vertex of the cable,* then:

$$g^1 y = 4 g^1 f \frac{x^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 51.$$

By placing w for the total uniform moving and dead loads per horizontal foot of the structure, including both the weight of

* That part of the suspenders lying below the vertex of the cable forms a uniform load over the span.

that part of the suspenders below the vertex of the cable and g_0, and by the aid of Eqs. 50 and 51, Eq. 13 of Art. 26 becomes:

$$H \cdot \frac{d^2 y}{d x^2} = w + \left(g_0 \frac{64 f^2}{l^4} + g^1 \frac{4 f}{l^2} \right) x^2 \quad . \quad . \quad 52.$$

Integrating Eq. 52 twice between the limits of x and 0, and placing $\frac{2}{3} \frac{f}{l^2} \left(16 \frac{f}{l^2} g_0 + g^1 \right) = k^1$:

$$y = \frac{x^2}{2H} \left(w + k^1 x^2 \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad 53.$$

in which H is the horizontal component of the cable stress. By placing $x = \frac{l}{2}$ and $y = f$ in Eq. 53:

$$H = \frac{l^2}{8f} \left(w + k^1 \frac{l^2}{4} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad 54.$$

This value of H will permit the resultant cable stress to be found at any point where the inclination is α, since that stress is:

$$T = H \sec \alpha = \frac{l^2}{8f} \left(w + k^1 \frac{l^2}{4} \right) \sec \alpha \quad . \quad . \quad 55.$$

If the length l^1 of the floor construction or stiffening-trusses supported by the cable is less than the length of span l of the cable, that difference in length may readily be allowed for, in finding the value of H, in the following manner:

Let the assumed uniform weight per linear foot of the length l^1 of the suspenders, the floor, and the stiffening-trusses, be represented by g_1. Then the centre bending moment in the span l produced by the symmetrically placed moving load $w_1 l^1$ concurrently with the fixed load $g_1 l^1$ will be:

$$M = (w_1 + g_1) \left(\frac{l^1}{2} \cdot \frac{l}{2} - \frac{l^1}{2} \cdot \frac{l^1}{4} \right) = (w_1 + g_1) \frac{l^2 - (l - l^1)^2}{8}.$$

Hence that part of H due to M is:

$$H'' = \frac{M}{f} = (w_1 + g_1) \frac{l^2 - (l - l^1)^2}{8f} \quad . \quad . \quad . \quad 56.$$

That part of H due to the weight of the cable is readily found from Eq. 52:

$$H^1 \frac{d^2 y}{d x^2} = g_0 \left(1 + \frac{64 f^2 x^2}{l^4} \right).$$

Two integrations, the first between x and o and the second between $\frac{l}{2}$ and o, give:

$$H^1 = g_0 \left(1 + \frac{8}{3}\frac{h^2}{l^2}\right)\frac{l^2}{8f} \quad . \quad . \quad . \quad . \quad . \quad 57.$$

The horizontal cable stress H will then take the value:

$$H = H^1 + H'' = g_0\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right)\frac{l^2}{8f} + (w_1 + g_1)\frac{l^2 - (l - l^1)^2}{8f} \quad . \quad 58.$$

Art. 30.—Weight of Cable and Suspenders.

Let F_0 be the area of cross-section of the cable in square inches at the vertex, γ the specific weight of the metal in lbs. per cu. inch, t the greatest permissible working stress in the cable in pounds per sq. in., then the weight per linear foot of the cable at the vertex will be, since $F_0 = \frac{H}{t}$:

$$g_0 = 12\, F_0\, \gamma = \frac{12\, H\, \gamma}{t} \quad . \quad . \quad . \quad . \quad . \quad 59.$$

Similarly, if w_0 is the total uniform load per linear foot of span, moving and fixed, carried by the suspenders (including that part of the latter below the lowest point of the cable), and if t^1 is the greatest permissible working stress in them, their weight per vertical foot for each foot of span will be:

$$g^1 = \frac{12\, w_0\, \gamma}{t^1}.$$

The substitution of g_0 and g^1 in Eq. 54, remembering that $w = w_0 + g_0$, gives:

$$H_{max} = \frac{w_0 l^2}{8f}\,\frac{1 + \frac{2f\gamma}{t^1}}{1 - \frac{3}{2}\frac{\gamma}{t}\frac{l^2}{f}\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right)} \quad . \quad . \quad . \quad . \quad 60.$$

An equation similar to Eq. 60 can also be readily established for a wire or other cable of uniform section. If t_0 is the intensity of tension in the cable at its lowest point of vertex,

$$H_{max} = F_0\, t_0.$$

The weight of the cable per linear foot is $12 F_0 \gamma$. It will be shown by Eq. 70 of this Article that the total length of the cable is approximately $l\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right)$. Hence the weight of the entire cable is the product of the preceding two quantities. Let w^1_1 represent the total dead and live load per linear foot of the whole structure except the cable whose weight is expressed above. Then by taking moments, the horizontal component of the cable stress, *i. e.*, the cable stress itself at the vertex is:

$$F_0 t_0 = \left[12 F_0 \gamma l\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right) + w^1_1 l\right]\frac{l}{8f},$$

$$\therefore F_0 t_0 = H_{max} = \frac{w^1_1 l^2}{8f}\,\frac{1}{1 - \frac{3}{2}\frac{\gamma}{t_0}\frac{l^2}{f}\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right)} \quad . \quad 61.$$

If it be remembered that w^1_1 includes the weight of the suspenders it will be seen that Eq. 61 is nearly identical with Eq. 60. If greater accuracy should be desired, the term $-\frac{32}{5}\frac{f^4}{l^4}$ from Eq. 21 can be added to the parenthesis in the denominator of the second member of Eq. 61.

The weight of one-half of the cable whose sectional area is proportioned to its total stress becomes by the aid of Eqs. 50 and 59:

$$G = \int_0^{\frac{l}{2}} g_x\, dx = H_{max}\frac{6\gamma l}{t}\left(1 + \frac{16}{3}\frac{f^2}{l^2}\right) \quad . \quad . \quad 62.$$

Eq. 51 shows that the weight of that part of the suspenders above the vertex of the cable, per linear foot of span, is $4 g^1 f \frac{x^2}{l^2}$. It is also shown immediately preceding Eq. 60 that $g^1 = \frac{12 w_0 \gamma}{t^1}$. The total weight of the suspenders for the half span, therefore, is:

$$\int_0^{\frac{l}{2}} 48 \frac{w_0 f \gamma}{t^1 l^2} x^2\, dx = \frac{w_0 l}{2}\frac{4 f \gamma}{t^1} \quad . \quad . \quad . \quad . \quad 63.$$

Hence the vertical component of the cable stress at the

extremity of the span (one-half of the total moving load and own weight of the entire structure) may be written:

$$V_{max} = \frac{w_0 l}{2}\left(1 + \frac{4fr}{l^2}\right) + G \quad . \quad . \quad . \quad . \quad 64.$$

The greatest stress in the cable, *i. e.*, at the points of support will be:

$$T_{max} = \sqrt{V^2_{max} + H^2_{max}} \quad . \quad . \quad . \quad . \quad 65.$$

The inclination of the cable at the points of support, α_m, is at once determined by the expression:

$$tan\ \alpha_m = \frac{V_{max}}{H_{max}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 66.$$

Art. 31.—Length of Cable or Chain.

The length, L, of the parabolic cable or chain may readily be determined by the aid of the equation of the parabola:

$$y = \frac{4f}{l^2}x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 67.$$

The length of the entire chain will be:

$$L = 2\int_0^{\frac{l}{2}}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \quad . \quad . \quad . \quad . \quad 68.$$

Taking the value of $\frac{dy}{dx}$ from Eq. 67:

$$L = 2\int_0^{\frac{l}{2}}\left(1 + \frac{64f^2}{l^4}x^2\right)^{\frac{1}{2}} dx \quad . \quad . \quad . \quad 69.$$

Developing the radical by the binomial formula to three terms, and then performing the integration indicated:

$$L = l\left(1 + \frac{8}{3}\frac{f^2}{l^2} - \frac{32}{5}\frac{f^4}{l^4}\right) \quad . \quad . \quad . \quad . \quad 70.$$

Greater accuracy could be attained by integrating more terms of the binomial series but it is not needed for this purpose. In fact, the third term of the second member of Eq. 70 is often neglected in approximate computations.

If $Y = 1 + \frac{8}{3}\frac{f^2}{l^2} - \frac{32}{5}\frac{f^4}{l^4}$, Eq. 70 may be written:

$$L = l\,Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 70a.$$

Practically all ratios of $\frac{f}{l}$ will be found between .2 and .05. The following table of values of Y may then be written:

$f \div l$	Y
.2	1.09643
.175	1.07566
.15	1.05676
.125	1.040104
.1	1.026027
.075	1.014797
.05	1.006627

These results are very near to those found by the exact formula in Art. 5.

Art. 32.—Approximate Greatest Length of Span.

An examination of the denominators of Eqs. 60 and 61 shows that for certain values of the working stress t and ratio $\frac{f}{l}$ the second part of that denominator will become zero, making the horizontal stress H_{max} infinitely great. The span length l so found is evidently the greatest possible for the given values of t or t_0 and $\frac{f}{l}$. Placing:

$$1 - \frac{3}{2}\frac{\gamma}{t}\frac{l^2}{f}\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right) = 0 \quad . \quad . \quad . \quad . \quad 71.$$

there will result:

$$l = \frac{\frac{2}{3}\frac{t}{\gamma}\frac{f}{l}}{1 + \frac{8}{3}\frac{f^2}{l^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad 72.$$

The value of l given by Eq. 72 is the greatest possible under the conditions of the analysis.

If the weight of a cubic inch of steel, γ, be taken as .2836 lb., the following table will give the greatest spans for the corresponding working stresses and ratios $\frac{f}{l}$:

Table 1. Approximate Greatest Lengths of Span.

	$f \div l = \frac{1}{9}$	$= \frac{1}{10}$	$= \frac{1}{12}$	$= \frac{1}{14}$	$= \frac{1}{16}$	$= \frac{1}{18}$
$t_0 = 27{,}500$	6,985 ft.	6,300 ft.	5,290 ft.	4,560 ft.	3,970 ft.	3,590 ft.
$t = 30{,}000$	7,620 ft.	6,870 ft.	5,770 ft.	4,970 ft.	4,360 ft.	3,920 ft.
$t_0 = 55{,}000$	13,970 ft.	12,600 ft.	10,580 ft.	9,110 ft.	7.990 ft.	7,190 ft.
$t = 60{,}000$	15,240 ft.	13,740 ft.	11,540 ft.	9,940 ft.	8,720 ft.	7,840 ft.

The value of $t = 60{,}000$ lbs. per sq. in. is a proper working stress for a steel wire cable, while $t = 30{,}000$ lbs. per sq. in., is suitable for a nickel steel eye-bar cable.

Art. 33.—Economical Ratio of Sag or Versine of Cable to Span.

The weights of the cable and suspenders, and hence their cost, as well as the weight and cost of the towers, depend directly on the sag or versine of the cable. If the effect of the height of the towers on the cost of the back stays and anchorage be neglected, the most economical ratio $\frac{f}{l} = n$ can readily be determined with close approximation.

Let R be the quotient of the total cost per foot of height of the two towers for one span, divided by the average cost per pound of the iron or steel of the suspenders and cables, so that $Rf = R \cdot n \cdot l$ will be the weight of metal whose cost is equal to that of the towers.

The weight of the suspenders above the elevation of the vertex of the cable for the entire span is shown by Eq. 63 to be

$$w_0 \, l \, \frac{4f\gamma}{t^1}.$$

The weight of the cable for the span is found directly by

the aid of Eqs. 62 and 60. The combined weights of the cable, the suspenders, and the equivalent of the towers then become:

$$\frac{\frac{3\,w_0\,l^2\,\gamma}{2\,t}\left(1+\frac{16}{3}n^2+\frac{2\,\gamma\,l}{t^1}\,n+\frac{32\,\gamma\,l}{3\,t^1}n^3\right)}{n-\frac{3\,\gamma\,l}{2\,t}\left(1+\frac{8}{3}n^2\right)}+w_0\frac{4\,\gamma\,l^2}{t^1}n+R\,l\,n=W.$$

The terms $\frac{32\,\gamma\,l}{3\,t^1}n^3$ and $\frac{3\,\gamma\,l}{2\,t}\left(1+\frac{8}{3}n^2\right)$ are each so small compared with the terms to which they are added or subtracted that they may be neglected without sensible error.

Hence the value of W becomes:

$$\frac{3w_0\,l^2\,\gamma}{2\,t}\left(\frac{1}{n}+\frac{16\,n}{3}+\frac{2\,\gamma\,l}{t^1}\right)+w_0\frac{4\,\gamma\,l^2}{t^1}n+R\,l\,n=W\quad .\quad 73.$$

Placing $\frac{d\,W}{d\,n}=0$ and solving for n:

$$n=\frac{1}{\sqrt{\frac{16}{3}+\frac{8\,t}{3\,t^1}+\frac{2\,R\,t}{3\,w_0\,l\,\gamma}}}\quad .\quad .\quad .\quad .\quad 74.$$

If, approximately, $l = 3{,}000$ feet; $t = 60{,}000$ lbs. per sq. in.; $t^1 = \frac{1}{2}\,t$; $w_o = 22{,}000$ lbs. per lin. ft.; and $R = 30{,}000$:

$$n=\frac{1}{\sqrt{\frac{32}{3}+64}}=\frac{1}{8.7},\ \text{nearly.}$$

This application of Eq. 74 shows how the economic ratio of $f \div l$ is affected by the cost of the towers.

Art. 34.—Deflection at Centre of Span Due to Deformation of Cable by Symmetrical Loading.

Let an unstiffened cable carry a symmetrical uniform moving load, w_1, per linear foot, the length of such load being l_1, as shown in Fig. 6. Also let the total dead or fixed load of the structure, including cable and suspenders, and assumed to be uniformly distributed over the span, be represented by g per

linear foot. The curve of the cable will then be a parabola. If l is the span and L the length of the cable in the same unit (*i. e.*, the foot), then by Eq. 70:

$$L = l\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right); \quad . \quad . \quad . \quad . \quad . \quad 75.$$

omitting the small third term of the second member of Eq. 70 without sensible error.

It will be further assumed that the cable or chain is per-

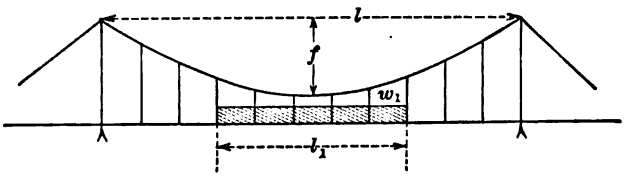

FIG. 6.

fectly flexible, and that the saddles or points of support at the tops of the towers are not moved by the loading.

The versine or sag f^1 of the cable after the load $w_1 l_1$ is applied is found by taking moments about the centre of span in the usual manner, H being the cable stress also at the centre of span:

$$f^1 = \frac{g\,l^2}{8\,H} + \frac{w_1\,l_1\,(l - \frac{1}{2}\,l_1)}{4\quad H} \quad . \quad . \quad . \quad . \quad 76.$$

The loaded and unloaded portions of the cable or chain are parts of different parabolas, but in which the horizontal stress H is the same. The versine f_1 of the loaded part may therefore be written:

$$f_1 = \frac{(w_1 + g)\,l_1^2}{8\,H}.$$

As the vertical component of the cable stress must be the same for both parts of the cable at the extremities of the loaded portion, the horizontal length of that part of the new cable curve, if it were a continuous parabola carrying the load g only, corresponding to l_1, is

$$l_1^1 = \frac{w_1 + g}{g}\,l_1 = \left(1 + \frac{w_1}{g}\right) l_1; \text{ and its sag is } f_1^1 = \frac{g\,l'^2_1}{8\,H}.$$

Similarly, the length of span l'' of that continuous cable curve carrying g, and its sag f'', will be:

$$l'' = (l - l_1) + \frac{w_1 + g}{g} l_1 = l + \frac{w_1}{g} l_1;$$

$$\text{and, } f'' = \frac{g\left(l + \frac{w_1}{g} l_1\right)^2}{8H}.$$

The length L of the cable in terms of parts produced by the loading will then be:

$$L = l''\left(1 + \frac{8}{3}\frac{f''^2}{l''^2}\right) - l_1'\left(1 + \frac{8}{3}\frac{f_1'^2}{l_1'^2}\right) + l_1\left(1 + \frac{8}{3}\frac{f_1^2}{l_1^2}\right) \quad 77.$$

Using the preceding values for l'_1, l'', f'_1, f'' and f_1 in the second member of Eq. 77, and equating the values of L in Eqs. 75 and 77:

$$l\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right) = l + \frac{1}{24H^2}\left(g^2 l^3 + 3 w_1 g l^2 l_1 + 3 w_1^2 l l_1^2 - (w_1 g + 2 w_1^2) l_1^3\right) \quad 78.$$

Hence:

$$H = \frac{1}{8f}\sqrt{l\left(g^2 l^3 + 3 w_1 g l^2 l_1 + 3 w_1^2 l l_1^2 - w_1(g + 2 w_1) l_1^3\right)}. \quad 79.$$

By combining Eqs. 76 and 79:

$$\frac{f'}{f} = \frac{g l^2 + w_1 l_1 (2 l - l_1)}{\sqrt{l\left(g^2 l^3 + 3 w_1 g l^2 l_1 + 3 w_1^2 l l_1^2 - w_1 (g + 2 w_1) l_1^3\right)}}. \quad 80.$$

This equation can be put in more convenient form by dividing both numerator and denominator of the second member by $g l_1$ and then subtracting unity from each member:

$$\frac{f' - f}{f} = \frac{\Delta}{f} = \frac{1 + \frac{w_1}{g}\left(2\frac{l_1}{l} - \frac{l_1^2}{l^2}\right)}{\sqrt{1 + \frac{3 w_1}{g}\frac{l_1}{l} + 3\frac{w_1^2}{g^2}\frac{l_1^2}{l^2} - \frac{w_1}{g}\left(1 + 2\frac{w_1}{g}\right)\frac{l_1^3}{l^3}}} - 1. \quad 81.$$

The desired centre deflection may readily be computed from Eq. 81.

That equation shows that $\Delta = 0$ for both $l_1 = 0$ and $l_1 = l$, as the deflection is due only to change of form of the curve.

There is some ratio of $l_1 \div l$, which will give a maximum value of Δ. That ratio can be found by placing equal to zero the first derivative of Δ with $l_1 \div l$ or l_1 as the variable. That process however leads to an equation of the fourth degree, and it is much more convenient to construct a diagram by the aid of Eq. 81 which will show the value of Δ for different values of $\frac{l_1}{l}$ and $\frac{w_1}{g}$. This can easily be done by using the following tabulated values of the centre deflections for $f = 1$. The centre deflections for other values of f are found by simply multiplying the values in the table by the proper value of f. By laying down the tabular quantities to a conveniently exaggerated scale curves for the various ratios, $\frac{w_1}{g}$ and $\frac{l_1}{l}$ will at once result. The greatest deflections given by the table are seen to occur for $l_1 = .2\, l$ when $\frac{w_1}{g} = 2$ or 3, and for $l_1 = .3\, l$ for the other ratios of $\frac{w_1}{g}$.

The centre deflections due to combined stretching and deformation of the cable are to be found by computing each part separately, using the value of f existing with the stretched cable by the methods set forth and adding the results.

Centre Deflections for $f = 1$.

$l_1 \div l$	$w_1 \div g$				
	.2	.5	1	2	3
.1	.0077	.019	.033	.0553	.071
.2	.0118	.026	.0445	.0667	.0798
.3	.01305	.028	.045	.063	.073
.4	.0126	.026	.0425	.054	.061
.5	.0106	.02	.032	.043	.048
.6	.0079	.016	.0234	.031	.0342
.7	.0051	.0104	.0149	.0195	.0217
.8	.00252	.0054	.0077	.0078	.0112
.9	.00075	.0013	.002	.0029	.0032
1.	.00	.00	.00	.00	.00

Art. 35.—Centre Deflection Due to a Change of Length of Cable.

The approximate length of cable as given by Eq. 70 will be used for this purpose. By differentiation of that equation, after placing $\frac{f}{l} = n$:

$$\Delta f = \frac{15}{16\,(5n - 24n^3)}\,\Delta L \quad . \quad . \quad . \quad . \quad 82.$$

The change of length of the cable between the towers may be produced by elastic extension, change of temperature, or by the strains in the masonry of the anchorages. If s is the uniform intensity of stress in the cables due to the moving load w_1 and dead load g per linear foot of span, the elastic stretch of the cable due to the total fixed or dead load g per linear foot will be:

$$\Delta L_g = \frac{s}{E}\frac{g}{g + w_1}L \quad . \quad . \quad . \quad . \quad . \quad 83.$$

Similarly, the elastic stretch due to the moving load will be:

$$\Delta L_1 = \frac{s}{E}\frac{w_1}{g + w_1}L \quad . \quad . \quad . \quad . \quad . \quad 84.$$

The total elastic stretch due to dead and live load combined will be

$$\Delta L_g + \Delta L_1 = \frac{s}{E}L \quad . \quad . \quad . \quad . \quad . \quad 85.$$

The change of length due to $\pm T^0$ change of temperature, if the coefficient of expansion per degree is C, will be:

$$\Delta L_t = \pm C T^0 L \quad . \quad . \quad . \quad . \quad . \quad 86.$$

There may be taken for the Centigrade scale, $C = .000{,}012{,}4$.
There may be taken for the Fahrenheit scale, $C = .000{,}006{,}9$.

If the cable extends over the towers (carrying fixed saddles at their tops) to anchorages, and if the cables slide over these saddles, then if l_1 is the length of the straight backstay span,

as shown in Fig. 7, Art. 37, there must be placed for L in Eqs. 83 to 86, both included:

$$L = l\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4\right) + 2\,l_1 \sec\alpha_1 \quad . \quad . \quad 87.$$

The angle α_1 is the inclination of the straight backstay to a horizontal line.

If a horizontal displacement of the saddle takes place, as is usually the case, its value will be closely:

$$\Delta L_e \sec\alpha_1 = \tfrac{1}{2}\,\Delta l; \quad . \quad . \quad . \quad . \quad . \quad 88.$$

in which L_e and $\sec\alpha_1$ are the change of length of backstay and the *secant* of its inclination to a horizontal line, respectively, while Δl is the change of length of the main or carrying span of the cable caused by the horizontal displacement at the top of the two towers. The deflection $(\Delta f)^1$ produced by this horizontal displacement will be found by differentiating Eq. 70, considering L constant and f and l the variables. That operation will give, remembering that $L \div l$ is given in terms of n by Eq. 70:

$$(\Delta f)^1 = -\frac{15 - 8\,(5n^2 - 36\,n^4)}{16\,(5\,n - 24\,n^3)}\,\Delta l \quad . \quad . \quad 89.$$

The negative sign indicates that Δf and Δl take place in opposite directions.

It is shown in Eq. 88 that $\Delta l = 2\,\Delta L_e \sec\alpha_1$; hence the total deflection will become:

$$\Delta f + (\Delta f)^1 = (\Delta f)_t = \frac{15}{16\,(5\,n - 24\,n^3)}\,\Delta L + \frac{15 - 8\,(5\,n^2 - 36\,n^4)}{8\,(5\,n - 24\,n^3)}\,\Delta L_e \sec\alpha_1 \quad . \quad . \quad 90.$$

If, to form a general expression, there be placed:

$$\Delta L = c\,L = c\,l\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4\right);$$

$$\text{and } \Delta L_e = c\,L_e = c\,l_1 \sec\alpha_1;$$

in which the values of c are shown in Eqs. 83 to 86, and if these values be substituted in Eq. 90:

$$(\Delta f)_t = \left\{\left(\frac{l}{2} - l_1 \sec^2\alpha_1\right)n + \frac{15 + 96\,n^4}{16\,(5\,n - 24\,n^3)}\,(l + 2l_1 \sec^2\alpha_1)\right\}c \quad . \quad 91.$$

Art. 36.—Deflection at the Centre of Span Due to Displacement of Saddles.

If the saddles are horizontally displaced while the cables do not slide on them, the centre deflection due to that displacement alone is, as has already been shown:

$$(\Delta f)^1 = \frac{15 - 8(5n^2 - 36n^4)}{16(5n - 24n^3)} \Delta l \quad . \quad . \quad . \quad 92.$$

The negative sign in Eq. 89 not being essential is omitted.

If the displacement of the saddles is accompanied by such a sliding of the cables over them that the total length of the cable between anchorages remains unchanged, the resulting centre deflection can be obtained by differentiating Eq. 87, remembering that $\Delta L = 0$ and that l, l_1 and f are the variables, while $2 \Delta l_1 = -\Delta l$:

$$(\Delta f)'' = -\frac{15 - 8(5n^2 - 36n^4) - 15 \sec \alpha_1}{16(5n - 24n^3)} \Delta l \quad . \quad 93.$$

In Eq. 93, as in Eq. 89, the negative sign of the second member indicates only that the deflection and change of span Δl take place in opposite directions.

If the cables or chains are secured directly to the tops of the towers so that under different conditions of loading the span is changed by the elastic motions of those tops, the resulting centre deflections of the cables will be given by Eq. 89. It is only necessary to have Δl in that equation represent the sum of the horizontal displacements of the tops of the two towers.

Art. 37.—Unsymmetrical Deflections of the Cable.

The greatest unsymmetrical deflections of a parabolic cable will be produced by continuous uniform loads extending from one end of the span to any point in it, as shown in Fig. 7. Let the length of the moving load w_1 from the right end of the span be z, while the distance x^1 from the left end of the span to that point of the cable which is horizontal (*i. e.*, where the deflection is greatest) is desired to be found. Two cases arise, one in which

the point of greatest deflection is in the unloaded part of the span, and the other when it is in the loaded part. The former case will be treated first.

By taking moments about any point of the unloaded part of the span:

$$M = \left(\frac{1}{2} g\, l + \frac{w_1 z^2}{2 l}\right) x - \frac{g\, x^2}{2} \quad . \quad . \quad . \quad . \quad 94.$$

The point of greatest sag of cable is at the point of greatest moment, as expressed by Eq. 94. Hence at that point:

$$\frac{d\,M}{d\,x} = \frac{1}{2} g\,(l - 2\,x) + \frac{1}{2} w_1 \frac{z^2}{l} = 0 \quad . \quad . \quad . \quad 95.$$

Therefore:

$$\frac{x^1}{l} = \frac{1}{2}\left(1 + \frac{w_1}{g}\frac{z^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad . \quad 96.$$

Eq. 96 will locate the point of greatest deflection or sag when it is in the unloaded part of the span.

If the length of moving load z is so extended that the point

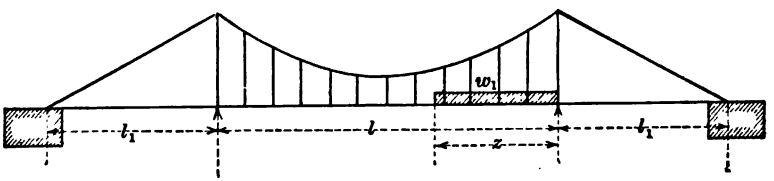

FIG. 7.

of greatest sag is in it, an expression for the bending about any point of z must be taken, x now being measured from the right-hand end of the span:

$$M = \frac{1}{2} g\, l\, x + \frac{w_1 z \left(l - \frac{z}{2}\right) x}{l} - \frac{(w_1 + g)\, x^2}{2} \quad . \quad 97.$$

Again, for the point where the cable is horizontal:

$$\frac{d\,M}{d\,x} = 0 = \frac{1}{2}\left\{g\, l + \frac{w_1 z\,(2\,l - z)}{l} - 2\,(w_1 + g)\, x\right\} = 0 \quad . \quad 98.$$

Hence: $$\frac{x^1}{l} = \frac{1}{2}\left\{\frac{1}{1+\frac{w_1}{g}} + \frac{1}{1+\frac{g}{w_1}}\left(2-\frac{z}{l}\right)\frac{z}{l}\right\} \quad . \quad . \quad 99.$$

Eq. 99 will determine the point of greatest sag of the cable when it is in the loaded part of the span, remembering that x^1 is measured, in this case, from the right-hand end of the span.

Both Eqs. 96 and 99 show that x^1 increases with z, hence, with the moving load as taken, the point of greatest sag of the cable first makes an excursion to the right at the limit of which that sag or deflection has its maximum value, and then as the length z of the moving load increases it moves back toward the centre of span. It follows from these considerations that the greatest sag occurs at the head of the moving load when the latter has the length $z^1 = l - x^1$, or when: $\frac{x^1}{l} = 1 - \frac{z^1}{l}$. The substitution of this value of $\frac{x^1}{l}$ in Eq. 96 gives

$$\frac{z^1}{l} = \frac{g}{w_1}\left(\sqrt{1+\frac{w_1}{g}} - 1\right) \quad . \quad . \quad . \quad . \quad 100.$$

Eq. 100 will give the proportionate part of the span to be covered with the moving load in order to produce (at the head of that moving load) the greatest possible unsymmetrical sag.

The greatest excursion of the lowest point of the cable from the centre of the span is:

$$\frac{e}{l} = \frac{1}{2} - \frac{z^1}{l} = \frac{1}{2} - \frac{g}{w_1}\left(\sqrt{1+\frac{w_1}{g}} - 1\right) \quad . \quad . \quad 101.$$

Neglecting the stretch of the cable, its versine while subjected to unsymmetrical deflections may be considered constant and equal to f without essential error. The middle point of the cable, however, will be raised or suffer a negative deflection $(-\Delta f)$ which may be taken equal to the versine of a parabola whose span is $2e$. Since the parabolas whose spans are $2e$ and $l + 2e$ have the same parameter:

$$(-\Delta f) = \left(\frac{2e}{l+2e}\right)^2 f = \left(\frac{2\frac{e}{l}}{1+2\frac{e}{l}}\right)^2 f \quad . \quad . \quad 102.$$

The following values can be computed by the aid of Eqs. 100, 101, and 102:

$\frac{g^1}{w_1}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	2	3	4	5
$\frac{s_1}{l}$	.309	.366	.395	.414	.45	.464	.472	.475
$\frac{e}{l}$	.191	.134	.105	.086	.05	.036	.028	.025
$\frac{\Delta f}{f}$	.0765	.0447	.0302	.0214	.0083	.0045	.00281	.00227

A diagram may readily be constructed from this table which would represent the results more satisfactorily.

The numerical results in the tabular statement are values for the length unity of span, *i. e.*, $l = 1$. For any other length of span, obviously, those values must be multiplied by that length.

Art. 38.—Secondary Stresses.

The preceding theory of the unstiffened suspension bridge is based on the assumption that the chain or cable is perfectly flexible. Under this assumption the resistance, due either to friction on the pins of the chain or to the stiffness of the cable, has been neglected. While the latter is too small in its influence on the stresses in the cable to be of any material consequence, it may be necessary to recognize the former. Observations made on existing chain suspension bridges by Steiner and Fraenkel, by means of the grapho-automatic apparatus of the latter, have shown the existence of no inconsiderable bending stresses in single members of the chain. If T represent the total direct or tensile stress in a bar of the chain, d the diameter of the pin, and ϕ the coefficient of friction, the bending moment produced in that bar by the friction may be taken as:

$$M = \phi T \frac{d}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 103.$$

Approximately, for the bar in greatest tension, $T = H\sqrt{1 + 16\frac{h^2}{l^2}}$; hence, if $\phi^1 = \phi\sqrt{1 + 16\frac{h^2}{l^2}}$, then $\phi\ T = \phi^1 H$. In order to find the maximum value of the bending moment, H must be determined for that condition of unsymmetrical loading which produces the greatest deformation of the cable and to which the moment of friction $\phi^1 H\frac{d}{2}$ corresponds.

The friction on the pins and the stiffness of the cable act like a stiffening-truss in preventing a little deformation. If this moment of friction were the greatest moment in a stiffening-truss produced by an unsymmetrical moving load w^1 per unit of span, it may have the approximate value $.0165\, w^1 l^2$. It may therefore be supposed that the actual moving load w_1 per linear foot is equal to $w^1 + w''$, the first being the unsymmetrical load described above, and the second, w'', a uniform load over the entire span. It may further be supposed that the horizontal stress H is due to the own weight g and w'' only. Hence, approximately

$$.0165\, w^1 l^2 = \phi^1 H\frac{d}{2} = \phi^1 .\frac{1}{8}(w'' + g)\frac{l^2}{f}\cdot\frac{d}{2},$$

$$\therefore w'' + g = \frac{.264\, w^1}{\phi^1 \cdot \frac{d}{f}} = \frac{.264\,(w_1 - w'')}{\phi^1 \cdot \frac{d}{f}} \quad . \quad . \quad 104.$$

Multiplying the first and third members of this equation by $\phi^1\frac{d}{f}$ and then adding $.264\,(w'' + g)$ to both sides:

$$w'' + g = \frac{w_1 + g}{1 + 3.79\,\phi^1 .\frac{d}{f}} \quad . \quad . \quad . \quad . \quad 105.$$

The greatest bending moment will then be:

$$M_p = .0165\, w^1 l^2 = \frac{\phi^1 d\, l^2}{f + 3.79\,\phi^1 d}\left(\frac{w_1 + g}{16}\right) \quad . \quad . \quad 106.$$

Frequently, and perhaps usually, $3.79\,\phi^1 d$ is very small in

comparison with f, and may be neglected. This approximation will then give:

$$M_p = \frac{\phi^1 d l^2 (w_1 + g)}{16 f} \text{ (very nearly)} = \phi^1 H \frac{d}{2} \quad . \quad . \quad 107.$$

These values of the bending moment will enable that quantity to be computed when the dimensions are given; or, on the other hand, they will enable the dimensions of the bars so to be determined as to allow for the bending stresses in addition to those of direct tension. In consequence of the probable corrosion between the pin and eye bar heads ϕ should be taken rather high and never less than 0.20.

Example. In an existing chain suspension bridge, the span $= l = 1000$ ft.; the versine $= f = 100$ ft.; g (for one chain) $=$ 3000 lbs. per lin. ft.; w_1 (for one chain) $= 1200$ lbs. per lin. ft.; and $d = .666$ ft. By computation $H = \frac{(w_1 + g) l^2}{8 f} = 5{,}250{,}000$ lbs. Hence, $M_p = \frac{.25 \times .666}{2} \times 5{,}250{,}000 = 437{,}500$ ft. lbs.

This moment is divided among 7 bars, each 10″ x 2″, and each carrying 62,500 ft. lbs. of bending moment. The apparent bending stress in the extreme fibres will therefore be 18,000 lbs. per sq. in., which is six-tenths of the direct tensile stress. The direct tension in the bar will reduce this bending, but the resultant greatest intensity can readily be computed by the formulæ for combined direct tension and bending in an eye bar.

CHAPTER IV

Statically Indeterminate Stiffening-Trusses

Art. 39.—Theory of the Straight Stiffening-Truss Based Upon the Elastic Deformation of the Structure.

The statical deformation of a flexible cable is usually prevented by connecting it with a straight or nearly straight truss, either partially or wholly below the cable, by means of parallel suspenders in its own plane. The truss so used is called a stiffening-truss. Its function is to distribute to the cable or frame, uniformly along a straight line, any load whatever applied to the structure, so that the parabolic form of the cable or frame shall be preserved under all conditions of loading.

It will be found that the stresses in the resulting stiffened structure are, in general, not statically determinate. As has already been shown, a simple assumption can be made which will render the stiffening-truss statically determinate with a fairly close degree of approximation. A more closely approximate theory of the stiffening-truss, based upon the elastic deformation of the structure, will next be given.

In both these theories it is assumed that there is no sensible variation from the parabolic form of cable or frame, statical or elastic, under any condition of loading.

Art. 40.—General Considerations.

Although the theory set forth in this chapter cannot be considered exact, for the reasons that it does not recognize any change in the parabolic form of the cable under different conditions of loading and that strictly accurate expressions for the work ex-

pended in the elastic deformations of the different parts of the structure are not used in all cases, yet it is sufficiently accurate for the design of long span suspension bridges with modern stiffening-trusses which prevent sensible deformation of the cable. Another nearly exact theory, based upon the equality between cable and truss deflections, will be set forth in a subsequent chapter.

Art. 41.—General Fundamental Formulæ.

The equations of Art. 25 and those of Art. 27 relating to the parabolic cable or frame are directly applicable to this case as they stand. The external forces P induce the moving load stresses s in the suspenders which are added to the panel loads K due to the weights of the trusses, suspenders, and cable. If the suspenders are placed the constant distance a apart along some assumed straight line, usually the horizontal span, Eq. 7 of Art. 25 will take the form:

$$-H\,\Delta^2 y_m = (S_{m-1} + K)\,a \quad . \quad . \quad . \quad . \quad 1.$$

If the corners of the polygonal frame lie in a parabola with its axis parallel to the suspenders and having the versine f and span l, with the suspenders equidistant from each other, it is obvious from the considerations set forth in Art. 27 that the stress in each suspender must be the same as that in any other, assuming concurrently that the weight of the cable is uniform per unit of span. Under the same conditions, if the suspenders are not equidistant from each other the loads which they carry will be in proportion to the distances between them. The function of the stiffening-truss is thus seen to be the distribution of the varying moving load equally to the suspenders.

Referring to Fig. 8 of Art. 42, the equation of the parabolic centre line of the cable when the origin of co-ordinates is at the top of one of the towers is:

$$y = \frac{4f}{l^2}(l - x)\,x.$$

If Δx is written for a in Eq. 1, that equation will give in

connection with the preceding expression for y:

$$H \frac{8f}{l^2} \Delta x = S_m + K \quad . \quad . \quad . \quad . \quad . \quad 2.$$

If the curve of the frame (arch) or cable is continuous, and if the load is vertical, the equations already found take simple forms. As its use has already indicated, K represents the weight per panel, a, of the frame, trusses, or cable, but in the case of the continuous curve that panel length is indefinitely short and equal to dx. For that reason if s_m is the intensity or amount of the moving load, suspender stress per unit of span $s_m\,dx$ must be substituted for S_m in Eqs. 1 and 2. Also, if w_0 is the intensity or weight per horizontal unit of the total fixed load or own weight of the structure and cable, $w_0\,dx$ must take the place of K. There will then result for the general case, Eq. 1:

$$s_m + w_0 = -H \frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad 3.$$

And for the parabolic curve, Eq. 2:

$$s + w_0 = H \frac{8f}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 4.$$

It will be assumed in the following theory that the equations 1 to 4, inclusive, hold true after deformation. In other words, it will be assumed that the elastic deformations of the truss and suspenders are so small that the uniformity of distribution of loading among the suspenders is not materially disturbed. This is more nearly true the more rigid the stiffening-truss is constructed.

This latter observation does not exclude the consideration of the effects of deformation of the cable or stiffening-trusses, or of the stretching of the suspenders. It is sometimes advisable to allow for the change of length of cable in the value of f, and again, it is usual to employ special formulæ to determine the effects of thermal or other deformations, all of which will be clearly indicated in the proper place.

Art. 42.—Bending Moments and Shears in the Stiffening-Truss.

Bending Moments.

The forces acting on the stiffening-truss are its weight (including all bracing, floor system, and flooring), the reactions, the moving load, and the upward pull of the suspenders. If the pull of the suspenders were absent the moments M_b and the shears S_b for any section x would be determined precisely as for an ordinary, simply supported beam, continuous or non-continuous as the case may be; or, in other words, such moments and shears would exist if there were no cable and the loads stated were wholly supported by the truss.

The stresses in the suspenders produce bending moments M_s in the trusses opposite, in general, to those caused by the vertical loading; hence, the resultant bending moment at any section of a stiffening-truss will be:

$$M = M_b - M_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 5.$$

In developing the following general equations for moments and shears it should be clearly observed that the entire own weight of the stiffening-trusses, including floor system and bracing, hangs wholly and directly on the cable, at mean temperature, through the action of the suspenders, thus causing neither bending moment nor shear at any point of the span. The moving load only causes such moments and shears. The effects of varying temperature will be treated in subsequent articles. In what follows in this chapter, therefore, the moment M_s will be considered as due to the uniform moving load suspender pull s'. Similarly the moment M_b will hereafter be treated as the ordinary bending moment in a non-continuous or continuous beam due to moving load only. This in no way whatever militates against the general principle expressed by Eq. 5.

Bending Moments for Non-Continuous Stiffening-Trusses.

In Fig. 8, let the straight line AB, drawn through the points of support A and B of the cable of a stiffened suspension bridge,

be taken as the axis of x with A as the origin, then let y be any ordinate of the cable. If w_0 is the (assumed) uniformly distributed weight of the stiffening-trusses, cable, and suspenders per unit of span, and if l is the length of the latter, the funicular polygon for the moving load stresses in the suspenders, combined with the weight of the trusses, suspenders, and cable, may be constructed in the usual manner from the following equation of moments:

$$M_s + \tfrac{1}{2} w_o x (l - x) = Hy \quad . \quad . \quad . \quad . \quad 6.*$$

If a funicular polygon next be constructed with the weights of the cable and suspenders added to the loads on the stiffening-truss as a simply supported beam, the resulting moments will

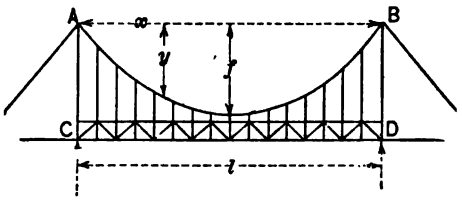

FIG. 8.

be represented by $M_b + \tfrac{1}{2} w_0 x (l - x) = Hy'$, in which y' is the ordinate of the polygon and H the same pole distance. The moment M_b may be composed, in general, of the uniform dead load moment and that of the moving load which may have any amount and distribution, but it is convenient to consider it due to moving load only. The substitution of M_b and M_s

* In all the following analysis relating to this method of treating the stiffened suspension bridge, the ordinate y of the cable, and especially its value f for the centre of the span, should, for accuracy, be given the value which belongs to the cable for the stressed condition due to the loads which it is carrying for any particular condition or case. If y or f be given their values for no loading in the formulæ to be established hereafter, a small error will exist for the reason that such values do not belong to what may be termed the loaded configuration of the cable. In order to give the formulæ their greatest accuracy, therefore, the central sag f should have the value which it takes for the particular condition of temperature or loading under which the formulæ are supposed to be applied.

from the two preceding equations in the value of M will give:

$$M = H(y' - y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 7.$$

This equation expresses the application of a well-known general principle in graphical statics; *the moment in the stiffening-truss is proportional to and may be represented by the vertical intercept between the equilibrium polygon of the actual loads (dead and moving) in the span and the actual frame (cable or arch)*, as shown in Fig. 8.

Bending Moments for Continuous Stiffening-Truss.

The same general form of equation or expression is equally applicable to a continuous stiffening-truss of three spans, *i.e.*, one centre span and two side spans.

Let M' and M'' represent the moments in the continuous truss at each end of a span *caused by the pull of the suspenders due to moving load*, then the ordinary theory of continuous beams* shows that Eq. 6 of this article will take the following form for the centre span, remembering that M_s is now the moment caused by the suspender pull in a continuous truss, so that M' and M'' must be affected by a negative sign:

$$M_s + \tfrac{1}{2} w_0 x(l - x) - M'' \frac{x}{l} - M' \frac{l - x}{l} = Hy.$$

It is convenient to write $M'' = Hm''$ and $M' = Hm'$, in which expressions m'' and m' are distances or lever arms whose signs may be the same as that of y or different from it, according as M'' or M' is a positive or negative bending moment. Hence the bending moment M_s in the continuous truss, due to the stresses in the suspenders, will become:

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*See "Resistance of Materials," by Wm. H. Burr, page 84, where

$$M_1 = M - M_a \frac{l - x}{l} - M_b \frac{x}{l}.$$

In this equation M_1 is identical with M_s of Eq. 6 above, and M with M_s in the two equations preceding Eq. 8. Also $M_a = M'$ and $M_b = M''$.

$$M_a = Hy - \tfrac{1}{2} w_0 x (l - x) + M'' \frac{x}{l} + M' \frac{l - x}{l} =$$

$$H \left(y + m'' \frac{x}{l} + m' \frac{l - x}{l}\right) - \tfrac{1}{2} w_0 x (l - x).$$

Hence, for the centre or main span:

$$\left.\begin{aligned} M &= M_b - Hy - M'' \frac{x}{l} - M' \frac{l - x}{l} + \frac{w_0 x}{2} (l - x) \\ M &= M_b - H\left(y + m'' \frac{x}{l} + m' \frac{(l-x)}{l}\right) + \tfrac{1}{2} w_0 x (l - x) \end{aligned}\right\} \quad . \quad . \quad 8.$$

For the side span, l_1, since the end moment is zero:

$$M = M_b - H \left(y_1 + m' \frac{x}{l}\right) + \tfrac{1}{2} w_0 x (l_1 - x). \quad . \quad 8a.$$

If x be measured from the left end of the span, for the span l_3:

$$M = M_b - H \left(y_3 + m'' \frac{l_3 - x}{l_3}\right) + \tfrac{1}{2} w_0 x (l_3 - x) \quad . 8a'.$$

The moments M'' and M', or the distances m'' and m', are readily computed by the aid of the theorem of three moments or constructed for the pole distance $H = 1$ for a given form of cable.

The Eqs. 8, 8a, and 8a' may be put into still simpler forms by observing that if H' is that part of H due to the moving load and H'' that due to the dead or fixed load, then $H = H' + H''$. Furthermore:

$$H'' y = \tfrac{1}{2} w_0 x (l - x).$$

A precisely similar equation holds for each of the side spans, l_1 and l_3. Hence, the values for M will take the following forms:

For the centre span:

$$M = M_b - H' \left(y + m'' \frac{x}{l} + m' \frac{l - x}{l}\right) \quad . \quad . \quad 8b.$$

For the side span, l_1:

$$M = M_b - H' \left(y_1 + m' \frac{x}{l_1}\right) \quad . \quad . \quad . \quad . \quad 8c.$$

For the side span, l_3:

$$M = M_b - H' \left(y_3 + m'' \frac{l_3 - x}{l_3}\right) \quad . \quad . \quad . \quad 8d.$$

In Eqs. 8*b*, 8*c*, and 8*d* the quantities m' and m'' will have the values $\frac{M'}{H'}$ and $\frac{M''}{H'}$ respectively:

As the dead load is known, the horizontal stress H'' may always be regarded as a known quantity.

In all the equations which follow, if all terms in which w_0 and k appear be cancelled (*i.e.*, be made zero), H' must be written in them instead of H.

Special Cases of Symmetrically Conditioned Spans.

Fig. 9 represents the case of the symmetrically conditioned continuous stiffening-truss carried over two side spans, but with no connection with the cables in the latter. Equal moments exist at the two piers, *i.e.*, at the bases of the two towers, and the only moments in the side spans are M_b and $M'' \frac{x}{l_1} = M' \frac{x}{l_1}$.

Hence, from Eqs. 8 and 8*a*:

For centre span: $M = M_b - H(y + m') + \frac{1}{2} w_0 x (l - x)$

For side spans: $M = M_b - Hm'' \frac{x}{l_1}$ 9.

If the curve of the cable is parabolic, or approximately if there is a parabolic polygon, the pull on the suspenders will be uniform or constant per linear unit of span. The truss, con-

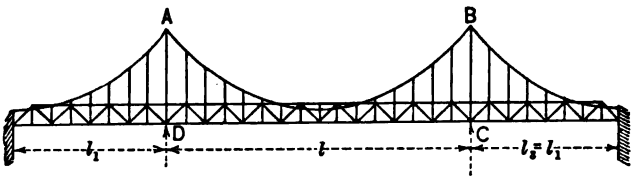

FIG. 9.

tinuous over three spans, will then be subjected to the uniform upward pull of the suspenders in the span l but to no suspender pull in the two side spans l_1.

In computing the end moments, $M'' = M'$, produced by the uniform upward pull of the suspenders, due to the moving

load only, careful consideration must be given as to the dead load of the trusses, including bracing and floor system. If the trusses in the centre span l are erected complete and attached to the cables by the suspenders, and if the side spans are also erected complete, but simply supported at their ends, and if the connections between the centre span and the end spans be then made at mean temperature the suspender pull in the centre span due to dead load will produce no bending either at the ends or at any other point of that span. This condition will be assumed in the present case, as it or an equivalent condition should be attained in the actual construction of such a bridge.

If M'' is the bending moment at either end of the span l, due to the uniform moving load pull s' on the suspender (the dead load of the trusses producing no bending moment at the ends of the span as explained in the preceding paragraph), the theorem of three moments gives:

$$M''(2l_1 + 3l) = -\frac{s'l^3}{4} \quad . \quad . \quad . \quad . \quad 9a.$$

If $r = \frac{l_1}{l}$ and if $H'f = \frac{s'l^2}{8}$, f being the sag of the cable, the preceding equation becomes, if m be written for m'':

$$\left.\begin{aligned} M'' &= -\frac{2H'f}{2r+3} = H'm \\ \therefore m &= -\frac{2f}{2r+3} \end{aligned}\right\} \quad . \quad . \quad . \quad 10.$$

Remembering that $H''y = \frac{1}{2}w_0 x(l-x)$, for the centre span, Eq. 9 will become:

$$\left.\begin{aligned} M &= M_b - H'\left(y - \frac{2f}{2r+3}\right) \\ &\text{For side spans:} \\ M &= M_b + \frac{2fH'}{2r+3}\frac{x}{l_1} \end{aligned}\right\} \quad . \quad . \quad . \quad 11.$$

Under the conditions to which Eqs. 11 apply, the moment M_b in the first of those equations is due wholly to the moving load, while in the second of those equations the moment M_b is due to the moving load combined with the dead load of the

complete trusses of the side spans considered as simply supported at their ends.

When the trusses in each of the side spans are connected by suspenders with the cable, as is the centre span, let the ordinate measured downward from the straight line connecting the extremities of the cable in that span, Fig. 10, be represented by y_1. The side spans will then be subjected to the uniform upward pull of the suspenders due to both dead and moving loads. Again, assuming the proper condition of the entire own weight of the spans and bracing to be carried directly and uniformly by the suspenders, in consequence of making the connections between the ends of the centre and side spans only after the complete trusses are erected and hung from the cables, the bending moments M'' at the ends of the centre span will be due wholly to the moving load. A similar application of the theorem of three moments to that already made will give for symmetrical conditions:

$$M''(2l_1 + 3l) + \frac{s'}{4}(l_1^3 + l^3) = 0.$$

Again representing the horizontal component of cable stress, due to the moving load only by H', and by s', the uniform suspender pull, due to moving load only, $H'f = \frac{s'l^2}{8}$. Placing $\frac{l_1}{l} = r$ the preceding equation will give, if m be written for m'':

$$\left.\begin{aligned} M'' &= -H'\frac{2(1+r^3)f}{2r+3} = H'm \\ \therefore\ m &= -\frac{2(1+r^3)f}{2r+3} \end{aligned}\right\} \quad . \ . \ . \quad 12.$$

Equations corresponding to Eqs. 9 then take the forms:

For centre span:

$$M = M_b - H'\left(y - \frac{2(1+r^3)f}{2r+3}\right)$$

For side spans:

$$M = M_b - H'\left(y_1 - \frac{2(1+r^3)f}{2r+3}\frac{x}{l_1}\right) \quad \left.\right\} \quad . \ . \quad 13.$$

In connection with the preceding equations the horizontal component H'' of the cable stress due to the dead load w_0 only may be at once computed when that dead load is known, as already indicated; hence it may be treated as a known quantity. The horizontal component H' of the cable stress due to moving load only will be determined later on.

The Ordinary Beam Moments M_b Due to Moving Loads.

The application of the preceding formulæ involves the use of the moments M_b, which may readily be determined for continuous trusses by the theorem of three moments. The formulæ for the general case shown in Fig. 10 will be given.

The moments of inertia of the normal sections of the trusses in the three spans l_1, l_2, and l_3 are I_1, I_2, and I_3. Furthermore, if I be written for I_2,

$$I_1 = r\,I_2 = r\,I \text{ and } I_2 = I = r'\,I_3.$$

The co-ordinate z locating a load in any span will be measured from the left end of each span. Then the following notation will be employed:

$$a = \frac{l_1}{l_2};\ b = \frac{l_3}{l_2};\ n = \frac{z}{l_1};\ m = \frac{z}{l_3}, \text{ and } q = \frac{z}{l_2}.$$

If M_2 is the moment over the support between spans l_1 and

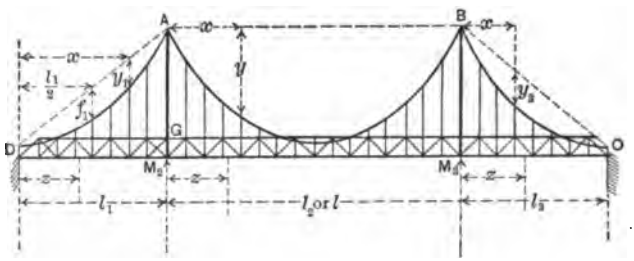

FIG. 10.

l_2, while M_3 is the moment between spans l_2 and l_3, remembering that the extreme end moments are zero, and if any loads P are

on any or all spans, the equations of the theorem of three moments show:

$$M_3 = l_2 \frac{a^2 \overset{1}{\Sigma} P(1-n^2)n - (2a+r) \overset{2}{\Sigma} P(1-q^2)q - 2b^2(a+r)r^1 \overset{3}{\Sigma} P(1-m^2)m}{4(a+r)(br^1+1)-r} \quad . \quad 14.$$

$$M_2 = -\, l_2 \frac{r\frac{M_3}{l_2} + a^2 \overset{1}{\Sigma} P(1-n^2)n + r \overset{2}{\Sigma} P(1-q^2)q}{2(a+r)} \quad . \quad . \quad 15.$$

In the ordinary case $l_1 = l_3$ and, hence, $a = b$. Also $I_1 = I_3$, and, hence, $r' = \frac{l}{r}$.

Let M_0 be the bending moment produced by any loads P on any span considered as a simply supported non-continuous span, then will the ordinary beam moment M_b take the following form for the span l_1:

$$M_b = M_2 \frac{x}{l_1} + M_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 16.$$

For the span l_2:

$$M_b = M_2 \frac{l_2 - x}{l_2} + M_3 \frac{x}{l_2} + M_0 \quad . \quad . \quad . \quad . \quad 17.$$

For the span l_3:

$$M_b = M_3 \frac{l_3 - x}{l_3} + M_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 18.$$

In each span x is measured from the left end.

Equations 14 to 18, both inclusive, enable every value of M_b to be found for every possible condition of loading.

If suspenders are in all three spans and if there is no moving load in any given span, M_0 will be zero for that span, and the corresponding term in Eqs. 14 and 15 will also be zero.

If there are no suspenders in the side or backstay spans, M_0 will be the moment produced by both the dead and moving load in those spans considered as simply supported at their ends, while M_0, for the centre span, will be the moment due to moving load only in that span. If, however, the side or backstay spans are so arranged as to carry no moving load, obviously M_0 will be caused by dead load only in those spans.

Finally, if the stiffening-trusses are non-continuous (*i.e.*, simply supported at their ends), $M_2 = M_3 = 0$ and $M_b = M_0$ for each span.

The Moments M′ and M″ Due to Suspender Pull.

The moments M' and M'' in the stiffening-trusses at the points of support between the centre and side spans caused by the uniform, upward, suspender pull s' per linear unit (foot) of span due to the moving load are readily written by the aid of Eqs. 14 and 15. In those equations, $P = s'\,dz$. Then, by observing the notation given above Eq. 14, there will result:

$$a^2 \overset{1}{\Sigma}\, P(1-n^2)\,n = \frac{a^2}{l_1^3}\,s' \int_0^{l_1} (l_1^2 - z^2)\,z\,dz = a^2 s' \frac{l_2}{4}$$

$$(2a+r) \overset{2}{\Sigma}\, P(1-q^2)\,q = \frac{s' l_2}{4}(2a+r).$$

$$2b^2(a+r)\,r' \overset{3}{\Sigma}\, P(1-m^2)\,m = 2b^2(a+r)\,r'\,s' \frac{l_2}{4}$$

$$r \overset{2}{\Sigma}\, P(1-q^2)\,q = r\,s' \frac{l_2}{4}.$$

Hence by Eq. 14:

$$M'' = \frac{s' l_2^2}{4} \frac{a^3 - 2a - r - 2b^2(a+r)\,r'}{4(a+r)(br'+1) - r} = s'B \quad . \quad 19.$$

And Eq. 15:

$$M' = -\frac{s' l_2^2}{8(a+r)} \left\{ \frac{r(a^3 - 2a - r - 2b^2(a+r)r')}{4(a+r)(br'+1) - r} + a^3 + r \right\} = -s'C \quad . \quad 20.$$

The values of B and C are obvious.

Ordinarily in the analysis which follows, l_2 is written as l, the main or centre span.

SHEARS

In order to determine the total transverse shear in the stiffening-truss, let S_b be the general expression for the corresponding shear in an ordinary truss, non-continuous or continuous, as the case may be, and carrying the same moving load as the stiffened suspension bridge, but on the supposition that the

suspenders carry no load. Also, as before, let α represent the angle between the cable at any point, and a horizontal line, while β is the angle between a horizontal line and the line connecting the summits of the cable or frame of a single span at the towers in the case of the latter having unequal heights. The angle β is positive when measured downward from a horizontal line as shown in Fig. 11. Suppose the cables to the right of B' and left of A' to be removed and the straight bar or strut $A'B'$ to be put in place. The latter acts on the stiffening-truss at E through the member $A'E$, while the reaction due to the pull of the suspenders only acts directly at E. The bar $A'B'$ would exert an upward thrust at B' and a downward thrust at A', each equal in amount to $H \tan \beta$. Each of these thrusts may be

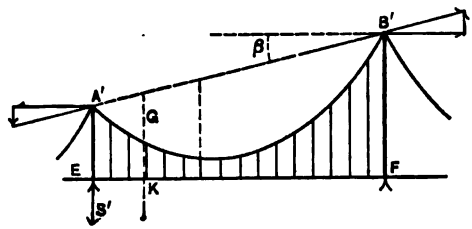

FIG. 11.

supposed to be displaced by a horizontal force H and the vertical component $H \tan \beta$, as shown in Fig. 11. The external vertical forces or shear, S_s, acting upon any part of the stiffening-truss, as $A'GKE$, and due only to the stresses in the suspenders, will then be:

$$-H \tan \beta - S' + s x = S_s$$

in which x is the distance EK and $-S'$ is the downward shear at E due to the upward (positive) pull of the suspenders. Remembering that $-S' + s x = H \tan \alpha$, the above expression for the vertical shear in the stiffening-truss due to the pull of the suspenders only becomes:

$$S_s = H \tan \alpha - H \tan \beta = H (\tan \alpha - \tan \beta) \quad . \quad 21.$$

This equation is general and H may include the horizontal cable stress due to the dead or fixed load. It will be more con-

venient, however, to make S_s represent the shear produced by the suspender pull s' due to moving load only, and to make the ordinary continuous or non-continuous truss shear S_b also due to moving load only. The horizontal cable stress H' caused by the moving load must then take the place of H. In the following equations, therefore, S_s, S_b, and H' are to be taken as indicated in this paragraph.

Observing that $H' m' = M'$ and $H' m'' = M''$, and remembering that S_b is now the moving load transverse shear at any section of an ordinary continuous or non-continuous beam, the following values for the shears in a straight stiffening-truss will result:

For a non-continuous truss over one span with towers of unequal height:

$$S = S_b - H' (\tan \alpha - \tan \beta)$$

If the towers are of equal height:

$$S = S_b - H' \tan \alpha \qquad \text{. . . 22.}$$

For a continuous truss with towers of unequal height, in general:

Remembering that in the case of an ordinary continuous beam, the shear at any point is $S_b + \frac{M'' - M'}{l}$; there may at once be written:

$$S = S_b - H' \left(\tan \alpha - \tan \beta + \frac{m' - m''}{l}\right)$$

If the towers are of equal height, in general: . 23.

$$S = S_b - H' \left(\tan \alpha + \frac{m' - m''}{l}\right)$$

The term $- H' \frac{m' - m''}{l}$ will be recognized as the shear transferred from one end of a continuous span to the other in consequence of the difference between the end moments M' and M'', as shown in the theorem of three moments.

If the end spans are of equal length the uniform suspender pull in all the spans will make $m' = m''$ and, hence, $\frac{m' - m''}{l} = 0$.

Adapting Eqs. 23 to the usual case of one centre or main

span and two side or backstay spans with equal height of towers, and remembering that as the equations are written the ordinary span shears S_b are to be computed from the left end of each span, there will result:

For side span l_1 (end moments zero):

$$S = S_b - H' \left(tan\, \alpha - tan\, \beta_1 - \frac{m'}{l_1} \right) \quad . \quad . \quad 24.$$

For centre span l_2:

$$S = S_b - H' \left(tan\, \alpha + \frac{m' - m''}{l_2} \right) \quad . \quad . \quad . \quad 25.$$

For side span l_3 (end moment zero):

$$S = S_b - H' \left(tan\, \alpha - tan\, \beta_3 + \frac{m''}{l_3} \right) \quad . \quad . \quad 26.$$

The value of the moments $M' = H\,m'$ and $M'' = H\,m''$ for every possible condition of loading are given by Eqs. 15 and 14, or 20 and 19.

If the side spans have no suspenders, but if they are continuous with the centre span, they may carry both dead and moving loads, which may materially affect the moments M_b in the centre span, the latter also carrying both dead and moving loads. These are important considerations and they should carefully be recognized in every design to which they apply.

Again, if the side spans have no suspenders and if the centre or main span be symmetrically conditioned, as indicated in Eqs. 9 to 11, it is only necessary to make $m' = m''$, or $m' - m'' = \theta$ in Eqs. 21 and 23.

Under these conditions the ends of the side spans may not need to be anchored down to their supports. This can be tested by covering the centre span with moving load and at the same time removing it entirely from the side spans. That condition of loading will show how much anchorage is required, if any, by comparison of the moving load reactions with the dead load reactions at the same points when no moving load is on any of the spans.

Art. 43.—Determination of the Horizontal Stress H.

In the preceding treatment of the stiffening-trusses the horizontal stress H is not determinable by purely static processes

unless the system of construction is such as to introduce special features or conditions which will remove the statical indetermination. Thus a constant horizontal stress might be given to the cable by making the anchorage a definite weight suspended from a lever or system of levers, so that the stiffening-truss would be subjected to an unchanging upward pull of the suspenders.

Another suggestion for the accomplishment of the same purpose is the "weighing" of the reactions by means of the simple lever arrangement shown in Fig. 12. If $v\,P$ is that part of

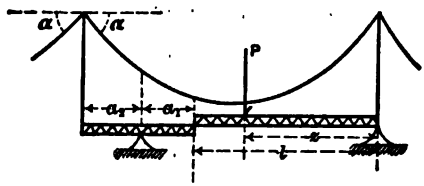

FIG. 12.

P carried by the stiffening-truss by the simple law of the lever while $(1 - v)\,P$ is carried to the cable by the suspenders, the conditions shown in Fig. 12 will give the following two equations:

$$2\,H\,tan\,\alpha\,a_2 = v\,P\,\frac{z}{l}a_1;$$

$$2\,H\,tan\,\alpha + v\,P = P.$$

In these equations H must be taken as due to the pull on the suspenders only.

If $\frac{a_1}{a_2} = n$, the elimination of v from the two preceding equations will yield:

$$H = \frac{P\,z}{2\,l}\,\frac{n}{1 + \frac{z}{l}n}\,cos\,\alpha \quad . \quad . \quad . \quad . \quad 27.$$

In this case the stiffening-truss will be subjected to the ordinary bending and shear of a beam loaded with $v\,P$ in addition

to the bending and shear due to its function as a stiffening-truss with its load $(1 - v)\,P$. Obviously, similar observations apply to a uniform load.

Again, statical determination can satisfactorily be attained by the introduction of a centre hinge in the stiffening-truss, since the bending moment M in that truss will then be zero for the centre section. If the moment at the centre of the span of all the load on the structure (the moving load corresponding to $(1 - v)\,P$), taken as if resting on a beam simply supported at each end, be represented by M_c, the versine or sag of the cable being represented by f, as heretofore, there will result:

$$H = \frac{M_c}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 27a.$$

or, for a continuous truss, in the case represented by Eq. 9, Art. 42:

$$H = \frac{M_c}{f - m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 27b.$$

The preceding methods of avoiding statical indetermination except by the use of a centre pin or hinge, are attended with too many practical difficulties to make them generally available. Hence, for the determination of H, recourse must be had to the elastic deformations of the system by means of some one of several methods. The change in the ordinates of the points of the cable added to the elongations of the suspenders can be made equal to the deflections of the corresponding points of the stiffening-truss, and the horizontal stress H can be determined from the resulting conditions, as has been done in Chap. V. Another method is the application of the "Theorem of Least Work of Deformation." This theorem is simply the application of the principle of "Least-Work" of elastic forces to this case. It is based upon the condition that the elastic deformations of a given structure by a given set of loads or forces must be such that the internal work performed in producing them will be the least possible. In this case the only elastic deformations to be considered are those produced by direct stresses (tension and

compression) and by bending moments; hence a general expression for the work performed in these two ways must be written.

Let P represent the direct stress acting along the axis of any piece, M the moment at any normal section of any single piece subjected to bending, L the length of any piece, F the area of a normal section of the piece, I the moment of inertia of any normal section of the latter about an axis through its centre of gravity, and E the coefficient of elasticity. Then the expression of the work performed by the tensile or compressive stresses of the entire piece will be:

$$X' = \int \frac{P^2}{2EF} dL^*.$$

In the case of the piece subjected to flexure, the work performed upon the entire member by bending it will be:

$$X'' = \int \frac{M^2}{2EI} dL^*.$$

The expression for the total work expended upon all the pieces or members of a structure for the production of elastic deformation, which must be a minimum, will then take the general form:

$$X = \Sigma X' + \Sigma X'' = \Sigma \int \frac{P^2}{2EF} dL + \Sigma \int \frac{M^2}{2EI} dL = min. \quad . \quad 28.$$

The problem of the stiffened suspension bridge, so far as this feature is concerned, is the determination of that horizontal stress H, which will make the expression for the work X a minimum, *i.e.*, X considered as a function of the variable H, is to be made a minimum. In the third member of Eq. 28, P and M are the only variables, since I is assumed to be constant for any piece or member. Hence, the analytic condition to be fulfilled and under which H is to be determined is:

$$\frac{dX}{dH} = \Sigma \int \frac{P}{EF} \frac{dP}{dH} dL + \Sigma \int \frac{M}{EI} \frac{dM}{dH} dL = 0 \quad . \quad . \quad 29.$$

*"Elasticity and Resistance of Materials," by Wm. H. Burr, pages 150 and 152.

In order to apply this general equation to the particular case considered it is necessary to express the direct stresses P and the bending moments M in terms of the known external forces and the unknown horizontal force H.

The case of a suspension bridge with the stiffening-truss continuous over the main and two side spans, as represented in Fig. 10 of Art. 42, will first be considered. The lengths of the side spans are l_1 and l_3.

The following notation will be employed:

l_1, l_2 or l and l_3 are lengths of side and main or centre spans.

F = area of cross-section of the cable or chain.

$F = F_0$ at the apex of the curve.

λ = length of an eye-bar of a chain taken as a side of a polygonal frame.

a = horizontal distance or panel length between two consecutive suspenders.

$2F_1$ = area of cross-section of stiffening-girder or truss (in an articulated truss, the cross-sectional area of two chords).

M = bending moment at any point in the stiffening-truss of any span (Eqs. 8 to 8d of Art. 42).

E_1 = coefficient of elasticity for towers or piers if they are not of steel. In such a case, in the following tabulation, for h' there must be written $\frac{E}{E_1} h'$.

I = moment of inertia of $2F_1$ about axis through centre of gravity for the main or centre span.

I_1 and I_3 = moments of inertia for the side spans l_1 and l_3, respectively.

$I_1 = r I$ and $I = r' I_3$.

F_2 = area of cross-section of a suspender for each unit of length of span.

y' = length of a suspender (*i.e.*, vertical projection of actual suspender if the cable is cradled).

y = length of ordinate of curve or polygonal frame measured from the chord connecting the supporting points of the cable or frame (See Fig. 10).

v = vertical distance of apex of cable above lower ends of suspenders. Hence $y' = f - y + v$.

h_s = height of stiffening-truss (between centres of gravity of chords).

f = versine of cable, *i.e.*, sag of cable at vertex.

h' = height of tower whose cross-sectional area is F_3.

h'' = height of tower from lower chord of stiffening-truss to top, or length of longest suspender.

α_1 = angle of inclination of the anchor or side-span cable to horizontal at top of tower.

α = angle of inclination of the main cable to horizontal at points of support or tops of tower.

α' = inclination of cable or frame to horizontal at any point.

ΣP = total moving load acting on the stiffening-truss = $w_1 L_1$ when length of uniform moving load w_1 is L_1.

A and B = reactions produced by ΣP with the stiffening-truss taken as an ordinary non-continuous or continuous beam.

w_o = total dead load of structure, including cable, per linear foot.

K = panel load produced by own weight of cable or frame, k being the uniformly distributed weight per linear unit. If panel length is a, $K = k\,a$.

s = suspender pull per linear unit (foot) of span due to dead and moving load.

s' = suspender pull per linear unit due to moving load only.

If it be considered that the stress in the cable or chain at any point is:

$$T_m = H\frac{\lambda}{a} = H \sec \alpha' \quad . \quad . \quad . \quad . \quad . \quad 30.$$

and if the cross-section of the cable is proportioned to the stress, then will $F = F_0 \frac{\lambda}{a} = F_0 \sec \alpha'$.

$$\text{Again, } \sec \alpha' = \frac{(d\,y^2 + d\,x^2)^{1/2}}{d\,x} = \left(\frac{d\,y^2}{d\,x^2} + 1\right)^{1/2} . \quad . \quad . \quad 30a.$$

It is assumed that the stiffening-truss is supported directly by the towers, at the elevation of the roadway, as that is the usual practice.

The general value of any single weight (moving load) in any span is P. This may be a moving panel load, or in case of a uniform moving load it may be $p\,d\,z$, p being the uniform load per linear unit (foot), as will be explained in the different cases arising.

The distance of P from the left end of any span will be represented by z. As shown in Fig. 10 the greatest versine or sag of the parabolic cable in the span l_1 will be f_1; it will be measured vertically downward from the centre of $A\,D$ or $B\,O$. The ordinate $A\,G = h''$ will represent the vertical distance of the point G in the lower chord of the truss below the supporting point, A, of the cable at the top of the tower. In the determination of the moment M_b, the abscissa x will be measured from the left end of each of the spans, l_1, l_2, and l_3.

If there are no suspenders in the side spans l_1 and l_3, the cross-section of the anchor cable or chain will be:

$$F' = F_0 \sec \alpha_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 31.$$

Eq. 29 must now be evaluated for the single members of the system, and for that purpose the following tabulation will be of service. In the table the stresses in the suspenders are taken from Eq. 7 of Art. 25, remembering that for a parabola $-\frac{\Delta^2 y}{a}$ is a positive quantity.

The parabolic curve will enable the following values to be written, taking the origin of co-ordinates at A or B, the top of a tower, or at D for the span l_1.

$$\left.\begin{aligned} y &= \frac{4f}{l^2}(l-x)\,x \\ y_1 &= \frac{4f_1}{l_1^2}(l_1-x)\,x \end{aligned}\right\} \quad \ldots\ldots \quad 32.$$

y_1 is measured vertically downward from the line $A\,D$ in Fig. 10, and x in the second equation is to be taken as a horizontal co-ordinate. If x were an inclined ordinate parallel to $A\,D$, it would be necessary to write $A\,D$ for l_1.

Eq. 32 shows that if a is taken as $d\,x$:

$$-\frac{\Delta^2 y}{a} = \frac{8f}{l^2}\,d\,x = \frac{8f}{l^2}\,a.$$

As s is the combined dead and moving load suspender pull per linear unit (foot) of span, except that of the cable, and as k is the corresponding weight of the cable:

$$s + k = H\frac{8f}{l^2}, \text{ or:}$$

$$s = \text{Suspender pull per unit of span} = H\frac{8f}{l^2} - k \quad . \quad 33.$$

If moving load alone is considered:

$$s' = H'\frac{8f}{l^2} \quad \ldots\ldots \quad 33a.$$

The work performed, therefore, per unit of span in stretching the suspenders, is:

$$\frac{1}{2F_2E}\left(H\frac{8f}{l^2} - k\right)^2 y'.$$

This expression must be multiplied by $d\,x$ and then integrated for the entire span. That operation will give:

$$\frac{1}{2F_2E}\int_0^l \left(H\frac{8f}{l^2} - k\right)^2 y'\,d\,x \quad \ldots\ldots \quad 34.$$

Referring to Eq. 29:

$$\frac{dX'}{dH} \text{ (for suspenders)} = \frac{1}{F_2 E}\int_0^l \left(H\frac{8f}{l^2} - k\right)\frac{8f}{l^2}y'\,dx \quad . \quad 35.$$

This expression is not strictly correct, for both k and y' vary with H, but that variation in $\frac{dX}{dH}$ is relatively small.

The Effect of Shearing in the Web on the Value of the Horizontal Stress H.

The expression for work expended in transverse and longitudinal shear may be written in two ways, *i.e.*, first, as if a solid plate web existed and sustained transverse and longitudinal shear like a solid beam with rectangular section, and second, for a web system composed of inclined or inclined and vertical compression and tension members as in ordinary stiffening-trusses of long span structures.

If h is the depth (or height) of the plate web of a stiffening-beam, b, its thickness, and if G is the modulus of elasticity, while S is the total transverse shear at any section, the total work of transverse and longitudinal shear for the entire span is:

$$X''' = \frac{6}{5G}\sum \frac{1}{b\,h}\int S^2\,dL \quad . \quad . \quad . \quad 35a.$$

The limits of the integration indicated belong to those parts of L over each of which $b\,h$ is constant. The general summation sign Σ simply shows that a summation of such integrations must be taken so as to cover the entire span. If the cross-section of the plate web is constant throughout the span the limits of the integral would be those of the span and the result would be:

$$X''' = \frac{6}{5\,b\,h\,G}\int^L S^2\,dL \quad . \quad . \quad . \quad . \quad 35b.$$

In order to provide for the work expended in producing shearing strains in a plate web or its equivalent, it will only be necessary to add to the second member of Eq. 28 one of the

expressions for X''' in Eqs. 35*a* and 35*b*. Also there would be added to the second member of Eq. 29:

$$\frac{dX'''}{dH} = \frac{12}{5G}\sum \frac{1}{bh}\int S\frac{dS}{dH}dL \quad . \quad . \quad . \quad 35c.$$

Or

$$\frac{dX'''}{dH} = \frac{12}{5bhG}\int_0^L S\frac{dS}{dH}dL \quad . \quad . \quad . \quad 35d.$$

The values of the shears S for the centre and side spans are given by Eqs. 21 to 26 of Art. 42. Remembering that $dH' = d(H - H'') = dH$, there may at once be written for the centre span, with towers of equal height and side spans of equal length, by Eq. 23:

$$\frac{dS}{dH} = -\tan\alpha = -\frac{4f}{l^2}(l - 2x) \quad . \quad . \quad . \quad 35e.$$

For either side span it is only necessary to give l in the parenthesis of the third member of this equation the proper subscript, *i.e.*, 1 or 3, and add $+\tan\beta$.

Inasmuch as the general value of the horizontal stress H, due to both moving and dead load, is to be found, and as $H''y = \frac{w_0}{2}(l - x)x$, there must be written:

$$-H\frac{4f}{l^2}(l - 2x) + \frac{w_0}{2}(l - 2x) \text{ for } -H'\frac{4f}{l^2}(l - 2x)$$

in the values of S indicated above.

Usually a mean cross-section of web may be determined with sufficient accuracy for the entire span, thus making it feasible to use Eq. 35*d*, and equations for that case will be developed.

If $m_2 = \dfrac{M_3 - M_2}{l}$ and if S_0 is the shear due to a load P at any point z in a simple non-continuous span, then, if $l_1 = l_3$ so that $m' = m''$, will the second of Eqs. 23 of Art. 42 give the general expression for the shear in the centre span written with H in place of H':

$$S = m_2 + S_0 + \frac{w_0}{2}(l - 2x) - H\frac{4f}{l^2}(l - 2x) \quad . \quad 35f.$$

Hence:

$$S\frac{dS}{dH} = -\left(m_2 + S_0 + \frac{w_0}{2}(l-2x)\right)\frac{4f}{l^2}(l-2x) + H\frac{16f^2}{l^4}(l-2x)^2 . \quad 35g.$$

Remembering that $dL = dx$, and observing that $\int_0^l (l - 2x)\,dx = 0$, and $\int_0^l (l-2x)^2\,dx = \frac{l^3}{3}$,

$$\int_0^l S\frac{dS}{dH}\,dx = -\frac{4f}{l^2}\left(\int_0^l S_0\,(l-2x)\,dx + \frac{w_0 l^3}{6}\right) + H\frac{16f^2}{3l} . \quad 35h.$$

If a single load P placed at the distance z from the left end of the span be taken, the shear for the part z of the span will be $S_0 = \frac{l-z}{l}P$, and for the part $l - z$, $S_0 = \frac{z}{l}P$. Hence:

$$-\frac{4f}{l^2}\int_0^l S_0\,(l-2x)\,dx = -4fP\left(1-\frac{2z}{l}\right)\left(1-\frac{z}{l}\right)\frac{z}{l} . \quad 35i.$$

By the aid of Eqs. $35h$ and $35i$, Eq. $35d$ will become:

$$\frac{dX'''}{dH} = -\frac{48f}{5bhG}\left\{P\left(1-\frac{2z}{l}\right)\left(1-\frac{z}{l}\right)\frac{z}{l} + \frac{l}{6}\left(w_0 - H\frac{8f}{l_2}\right)\right\} . \quad 35j.$$

It should be observed that $w_0 - H\frac{8f}{l^2} = -H'\frac{8f}{l^2}$.

As the load P may be placed at any point z in the span, Eq. $35j$ is perfectly general and suffices for any loading whatever.

The second member of Eq. $35j$ should now be placed in the second member of Eq. 29, and its terms will then appear in the last column of the tabulation following Eq. 35 under the heading $E\frac{dX}{dH}$. G, however, in Eq. $35j$ should be displaced by $nE = \frac{G}{E}E$.

This procedure will introduce the term $+F_0\frac{48f}{5bhn}\left(\sum P\left(1-\frac{2z}{l}\right)\left(1-\frac{z}{l}\right)\frac{z}{l} + \frac{w_0 l}{6}\right)$ into the numerator of Eq. 36 and the term $+ F_0\frac{64}{5bhn}\frac{f^2}{l}$ into the denominator of the same equa-

tion. If there is but one load P applied to the truss, the summation sign Σ should be omitted.

The preceding method may properly be used for an ordinary stiffening-truss by assuming a web plate approximately equivalent in its shearing capacity to the web members of the truss.

If desired, however, a form of expression for the work performed in resisting shear and adapted to individual web members may easily be written for a truss of uniform depth.

If S is the vertical shear in the truss at the point x, and if α is the angle which the web member at the same point (in tension or compression) makes with a vertical line, the total stress in that web member will be:

$$S \sec \alpha.$$

The expression X' (Eq. 28), for the work expended in stressing this member, the area of whose normal cross-section is F, is:

$$\int \frac{S^2 \sec^3 \alpha}{2EF} dL.$$

Hence:

$$\frac{dX'}{dH} = \int \frac{S \sec \alpha}{EF} \frac{d(S \sec \alpha)}{dH} dL.$$

Obviously $d(S \sec \alpha) = \sec \alpha \, dS$. Hence, for all the web members of the truss:

$$\frac{dX'}{dH} = \sum \frac{\sec^2 \alpha}{E} \int \frac{S}{F} \frac{dS}{dH} dL \quad . \quad . \quad . \quad 35k.$$

Usually the cross-section F of the member is constant throughout its length, and in that case F will be written outside of the integral sign.

Remembering that L, the length of the web member, is wholly independent of S, if F is constant for each member the

integral sign applies to dL only, and there is to be written $\int dL = L$. Eq. 35k then takes the form:

$$\frac{dX'}{dH} = \frac{1}{E}\sum \frac{L \sec^2 \alpha}{F} S \frac{dS}{dH} \quad . \quad . \quad . \quad . \quad 35l.$$

If L and α are uniform for a number, or even all, of the web members, $L \sec^2 \alpha$ would come outside of the summation sign Σ for such members.

The quantity $S\frac{dS}{dH}$ for each value of x to be taken is given by Eq. 35g. The summation is to be made for the entire span for each single load P or such other load as may be used in finding H.

After the terms produced by using Eqs. 35l and 35g in Eq. 29 are introduced in the tabulation below, it will be found that the quantity:

$$+F_0 \sum_0^l \frac{L \sec^2 \alpha}{F}\left(m_2 + S_0 + \frac{w_0}{2}(l - 2x)\right)\frac{4f}{l^2}(l - 2x)$$

must be added to the numerator of Eq. 36, and

$$+F_0 \sum_0^l \frac{L \sec^2 \alpha}{F} \frac{16 f^2}{l^4}(l - 2x)^2$$

to the denominator of the same equation.

The influence of the web members on the value of the horizontal stress H may be treated by either of the preceding methods, the first, perhaps, being the simplest.

It will be found, however, that there is no material error in ignoring the effect of the web members and the formulæ for H in this chapter will be written on that basis. If necessary the effect of the web for any special case can easily be introduced, as indicated above.

The preceding analysis is applicable precisely as it stands to

the side spans by simply using the proper subscript and, first, making:

For span l_2, $m_2 = \frac{M_2}{l_1}$;

For span l_3, $m_3 = -\frac{M_3}{l_3}$.

Then there must be written $-H\left(\frac{4f}{l^2}(l-2x)-\tan\beta\right)$ for $-H\frac{4f}{l^2}(l-2x)$, in which $\tan\beta = \frac{f}{l_1}$.

*Tabulated Statement from which H is Found.**

The tabulated statement for finding H may now be written as follows, remembering that in finding $\frac{dM}{dH}$ M'' and M' are taken from Eqs. 19 and 20 and that $s = H\frac{8f}{l^2} - k$, also that $s' = H'\frac{8f}{l^2}$. Finally, since $H = H' + H''$, H'' being constant, there may be written:

$$dH = dH'.$$

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* *Tabulated Statement for Non-Continuous Spans.*

The tabulated statement which follows is written in the most general form for continuous stiffening-trusses. If it should be desired to make this tabulation for non-continuous stiffening-trusses and thus avoid the additional work required in the general case, it can be simply done by omitting all terms in which M', M'', B, and C occur, as such terms are caused by the condition of continuity of the trusses. If the tabulated statement be written as it stands, but with the omission of the terms indicated, and if the solution for the horizontal component H of the cable stress be made precisely as is done in the text, Eq. 70 will at once follow. By this means the analysis required by the condition of continuity of trusses will be entirely eliminated.

Part	P or M	$\frac{dP}{dH}$ or $\frac{dM}{dH}$	F or I or I_1	$\int dL$	$E\frac{dX}{dH}$
Main Cable	$H\frac{\lambda}{a}=H\sec\alpha^1$	$\frac{\lambda}{a}=\sec\alpha^1$	$\frac{\lambda}{a}F_0=F_0\sec\alpha^1$	$\int dx\sec\alpha^1$	$\frac{H}{F_0}\int_0^l\left(\frac{dy^2}{dx^2}+1\right)dx$
Side Cables	$H\frac{\lambda_1}{a}=H\sec\alpha_1{}^1$	$\frac{\lambda_1}{a}=\sec\alpha_1{}^1$	$\frac{\lambda_1}{a}F_0=F_0\sec\alpha_1^1$	$\int dx\sec\alpha_1^1$	$\frac{H}{F_0}\int_{0\,0}^{l_1\,l_1}\left(\frac{dy_1{}^2}{dx^2}+1\right)dx$
Suspenders *	$\left(H\frac{8f}{l^2}-k\right)dx$	$\frac{8f}{l^2}dx$	$F_2\,dx$	y^1	$\frac{1}{F_2}\int_0^l\left(H\frac{8f}{l^2}-k\right)\frac{8f}{l^2}y^1\,dx$
	$\left(H\frac{8f_1}{l_1{}^2}-k\right)dx$	$\frac{8f_1}{l_1{}^2}dx$	$F_2\,dx$	$y^1{}_1$	$\frac{1}{F_2}\int_{0\,0}^{l_1\,l_1}\left(H\frac{8f_1}{l_1{}^2}-k\right)\frac{8f_1}{l_1{}^2}y^1{}_1\,dx$
					In the second of these integrals the origin of coordinates is taken at the vertex (lowest point) of the cable. For the usual cases ($l_1=\frac{1}{2}l$) this makes the double integral for span l_1 essentially equal to the integral for span l, and it will so be considered hereafter.
Two Towers	$2H(\tan\alpha+\tan\alpha_1)$	$2(\tan\alpha+\tan\alpha_1)$	$2F_2$	l^1	$\frac{2H}{F_2}h^1(\tan\alpha+\tan\alpha_1)^2$
Stiffening-Trusses; Centre Span	$M_b+\frac{w_0x}{2}(l-x)$ $-M''\frac{x}{l}-M'$ $\frac{(l-x)}{l}-Hy$ (See Eqs. 19 and 20)	$-y-\frac{8f}{l^2}\left[(B+C)\frac{x}{l}-C\right]$	I	$\int dx$	$\frac{1}{I}\left\{H\int_0^l y^2\,dx-\int_0^l M_b\,y\,dx+\frac{1}{l}\left(\int_0^l M''\,y\,x\,dx+\int_0^l M'(l-x)\,y\,dx\right)\right.$ $-\frac{w_0}{2}\int_0^l y\,x\,(l-x)dx+H\frac{8f}{l^2}\left((B+C)\int_0^l y\,x\,dx-Cl\int_0^l y\,dx\right)-$ $-\frac{8f}{l^2}\left((B+C)\int_0^l M_b\,x\,dx-Cl\int_0^l M_b\,dx\right)+\left(H\frac{64f^2}{l^4}-\frac{8fk}{l^2}\right)\int_0^l\left[(B+C)\frac{x}{l}\right.$ $\left.\left.-C\right]^2dx-\frac{w_0}{2}\frac{8f}{l^2}(B+C)\int_0^l x^2(l-x)\,dx+\frac{w_0}{2}\frac{8f}{l^2}C\int_0^l x\,(l-x)\,dx\right\}$
Side Spars *	$M_b+\frac{w_0x}{2}(l_1-x)$ $-M'\frac{x}{l_1}-Hy_1$	$-y_1$ $+\frac{8f}{l^2}C\frac{x}{l_1}$	I_1	$\int dx$	$\frac{1}{rI}\left\{H\int_{0\,0}^{l_1\,l_1}y_1{}^2\,dx-\int_{0\,0}^{l_1\,l_1}M_b\,y_1\,dx+\frac{1}{l_1}\int_{0\,0}^{l_1\,l_1}M'\,y_1\,x\,dx-\frac{w_0}{2}\int_{0\,0}^{l_1\,l_1}y_1x(l_1-x)dx\right.$ $-H\frac{8f_1}{l_1{}^2}C\int_{0\,0}^{l_1\,l_1}y_1\,x\,dx+\frac{8f}{l^2l_1}C\int_{0\,0}^{l_1\,l_1}M_b\,x\,dx+\left(H\frac{8f}{l^2}-k\right)\frac{8f}{l^2l_1{}^2}C^2\int_{0\,0}^{l_1\,l_1}x^2\,dx$ $\left.+\frac{w_0}{2}\frac{8f}{l^2l_1}C\int_{0\,0}^{l_1\,l_1}x^2(l_1-x)\,dx\right\}$
Anchor Cable or Backstay	$H\sec\alpha_1$	$\sec\alpha_1$	$F_0\sec\alpha_1$	$2l_1\sec\alpha_1$	$\frac{2H}{F_0}l_1\sec^2\alpha_1$

The effect of the moving load reactions on that part of the towers below the points of support of the stiffening-trusses is so small as to be negligible and, consequently, it is neglected in the preceding tabulation.

Remembering the preceding observations, also that differentials are to be used, and hence that $a = dx$, while λ and λ_1 are differentials of cable curves in the spans l and l_1, there may at once be written:

$$2\int_0^{l_1}\left(\frac{dy_1^2}{dx^2}+1\right)dx = 2\,l_1\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)$$ (See note on Page 116.)

$$\int_0^{l}\left(\frac{dy^2}{dx^2}+1\right)dx = l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)$$

$$\int_{0\,0}^{l_1\,l_1} y_1 x(l_1-x)\,dx = 2\frac{4f_1}{l_1^2}\int_0^{l_1}(l_1-x)^2x^2\,dx = \frac{4f_1l_1^3}{15} = \frac{4}{15}\frac{f_1}{l_1}l_1^4$$

$$\int_{0\,0}^{l_1\,l_1} y_1^2\,dx = \frac{16f_1^2l_1}{15} = \frac{16}{15}\frac{f_1^2}{l_1^2}l_1^3;\ tan\,\beta = \frac{h''}{l_1}$$

$$\frac{1}{l}\int_0^l yx\,dx = \frac{fl}{3};\ \frac{1}{l}\int_0^l (l-x)y\,dx = \frac{fl}{3};\ \frac{2}{l_1}\int_0^{l_1} y_1x\,dx = \frac{2f_1l_1}{3}.$$

$$y' = h'' - \frac{4f}{l^2}(l-x)\,x;\ \int_0^l y'\,dx = l\left(h''-\frac{2f}{3}\right).$$

$$y'_1 = \frac{h''}{l_1}x - y_1 = \left(h''-\frac{4f_1}{l_1}(l_1-x)\right)\frac{x}{l_1}.$$

$$2H\frac{64f_1^2}{l_1^4}\int_0^{l_1} y'_1\,dx = H\frac{128f_1^3}{l_1^3}\cdot\left(\frac{h''}{2f_1}-\frac{2}{3}\right).\ 2k\frac{8f_1}{l_1^2}\int_0^{l_1} y_1'\,dx$$

$$= 2\cdot k\frac{8f_1^2}{l_1}\left(\frac{1}{2}\frac{h''}{f_1}-\frac{2}{3}\right)$$

$$H\int_0^l y^2\,dx = H\frac{8}{15}f^2l = H\frac{8}{15}\frac{f^2}{l^2}l^3\cdot H\int_{0\,0}^{l_1\,l_1} y_1^2\,dx$$

$$= H\frac{16}{15}f_1^2l_1 = H\frac{16}{15}\frac{f_1^2}{l_1^2}l_1^3$$

*The double limits of integration indicate that the integration covers two equal side spans. If the latter are unequal the same integration must be applied to each of the side spans l_1 and l_2, for which also $I_1 = r\,I$ and $I = r'\,I_2$.

Again, if the modulus of elasticity for the suspenders is $E' = r'\,E$, there must be written $r'\,F_2$ for F_2 in the above expressions. Suspenders are frequently of steel cables for which it is sometimes assumed $r' = \frac{1}{2}$.

$$\frac{w_0}{2}\int_0^l y\,x\,(l-x)\,dx = w_0\frac{f\,l^3}{15} = \frac{w_0}{15}\frac{f}{l}\,l^4. \quad \frac{w_0}{2}\int_{0\,0}^{l_1\,l_1} y_1\,x\,(l_1-x)\,dx = \frac{2\,w_0}{15}f_1\,l_1^3 = \frac{2\,w_0}{15}\frac{f_1}{l_1}l_1^4.$$

$$tan\,\alpha = \frac{4f}{l};\quad tan\,\alpha_1 = \frac{4f_1}{l_1};\quad l_1 = n\,l.$$

In the case of the side spans having their stiffening-trusses carried by suspension rods it will be supposed that the side cables are parts of parabolas with the same parameter as the parabola of the main cable, and the value of y_1 above is written on that condition. In this case usually $\alpha = \alpha_1$ and $tan\,\alpha = tan\,\alpha_1$.

If, on the other hand, there are no suspenders in the side spans it will be supposed that the side cables or backstays are carried straight from the supporting point T to the anchorage, in which case $tan\,\alpha_1 = tan\,\beta$.

The values of M' and M'' are taken from Eqs. 19 and 20 of Art. 42, in which $s = H\frac{8f}{l_2} - k$, as shown by Eq. 33.

By placing the sum of the quantities $E\frac{dx}{dH}$ in the last column of the tabulation equal to zero and then introducing the values established above, the following value of H will result:

$$H = \frac{\frac{F_0}{F_2}k\left\{\frac{8f^2}{l}\left(\frac{h''}{f}-\frac{2}{3}\right)+\frac{16f_1^2}{l_1}\left(\frac{1}{2}\frac{h''}{f_1}-\frac{2}{3}\right)\right\}+\frac{F_0}{I}\left\{\int_0^l M_b y\,dx\right.}{l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)+2\,l_1\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)+\frac{F_0}{F_2}\left\{\frac{64\,f^3}{l^3}\left(\frac{h''}{f}-\frac{2}{3}\right)\right.}$$

$$\frac{+\frac{w_0}{15}\frac{f}{l}\,l^4+\frac{1}{r}\left(\int_{0\,0}^{l_1\,l_1}M_b\,y_1\,dx+\frac{2\,w_0}{15}\frac{f_1}{l_1}l_1^4\right)+\frac{8f}{l^3}\left\{(B+C)\int_0^l M_b x\,dx\right.}{+\frac{128f_1^3}{l_1^3}\left(\frac{1}{2}\frac{h''}{f_1}-\frac{2}{3}\right)\right\}+\frac{F_0}{F_2}2\,h'\,(tan\,\alpha+tan\,\alpha_1)^2+\frac{F_0}{I}\left(\frac{8}{15}f^2\,l\right.}$$

$$\frac{\left.-C\,l\int_0^l M_b\,dx\right\}-\frac{8f}{rl^2l_1}C\int_{0\,0}^{l_1\,l_1}M_b\,x\,dx\Big\}+\frac{F_0}{3I}\left\{(B-C)fl-\frac{2f_1l_1}{r}C\right.}{+\frac{16}{15r}f_1^2\,l_1+\frac{16}{3}\frac{f}{l}f\,(B-C)-\frac{32}{3r}\frac{f_1}{l_1}f_1\,C+\frac{64}{3}\frac{f^2}{l^3}\left(B^2-C\,(B-C)\right)}$$

$$+\frac{8f}{l}\left(B^2-C(B-C)\right)+\frac{16f}{rl^2}l_1C^2\Big\}-\frac{F_0w_0}{I}\Big\{\frac{2fl_1^3}{3rl^2}C-\frac{fl}{3}(B-C)\Big\}$$
$$+\frac{128f^2}{3rl^4}l_1C^2\Big) \qquad 36.$$

In this equation, $(tan\,\alpha + tan\,\alpha_1)^2 = \frac{16}{l^2}\left(f+\frac{f_1}{n}\right)^2$.

If $tan\,\alpha = tan\,\alpha_1 = \frac{4f}{l}$, as is frequently the case:

$$\frac{F_0}{F_2}\,2h'\,(tan\,\alpha + tan\,\alpha_1)^2 = \frac{F_0}{F_2}\,\frac{128\,h'f^2}{l^2} \quad . \quad . \quad 37.$$

Eq. 36 has been written as if the two side spans were of the same length, which is the usual case. If, however, the spans l_1 and l_2 are not the same, it is only necessary to write for each term of Eq. 36, containing l_1, two such identical terms except that l_2 will be written for l_1 in one of those terms and the numerical coefficient of each will be half of that of the original term. As examples, for $\frac{16f_1^2}{l_1}\left(\frac{1}{2}\,\frac{h''}{f_1}-\frac{2}{3}\right)$ there would be written $\frac{8f_1^2}{l_1}$ $\left(\frac{1}{2}\,\frac{h''}{f_1}-\frac{2}{3}\right)+\frac{8f_2^2}{l_2}\left(\frac{1}{2}\,\frac{h''}{f_2}-\frac{2}{3}\right)$; also for $\frac{1}{r}\int_{00}^{l_1 l_1} M_b\,y_1\,dx$ there would be written $\frac{1}{r}\int_0^{l_1} M_b\,y_1\,dx+\frac{1}{r^1}\int_0^{l_2} M_b\,y_2\,dx$. Other terms containing l_1 would be replaced in precisely the same manner.

Eq. 36 is perfectly general, the value of I being a correct mean established wherever advisable by a method shown below.

If w_0 and k be made zero in Eq. 36, the first member becomes H', *i.e.*, the horizontal component of the cable stress due to moving load only. This value H' will usually be needed.

It should also be observed that the horizontal component H'' of the dead load cable stress at mean temperature is to be computed by assuming the dead load as uniform and hanging directly on the cable.

If the towers are not of steel, but of a material whose modulus of elasticity is E_1, it is only necessary, as was stated on page 106, to write $\frac{E}{E_1}h'$ for h' in Eq. 36.

The Moment of Inertia I.

It is not difficult to determine with sufficient approximation the mean value I of the moment of inertia of the cross-section of the stiffening-truss, assuming that section to vary in all the spans. Remembering that the chord sections only enter the moment of inertia, and taking d as the depth of the truss or girder, the area of the upper or lower chord section of which is F_1, there may be written for any point:

$$I' = 2.F_1.\frac{d^2}{4} = F_1\frac{d^2}{2} \quad . \quad . \quad . \quad . \quad . \quad 38.$$

By using the approximate methods of Chapter I, especially if other designs of stiffening-trusses or girders are available for reference, reasonably close values of F_1 may be estimated for use in the formulæ of this chapter. In fact, it may easily be practicable to estimate in that way without further procedure a mean value of I close enough for use in Eq. 36. If this should not be considered satisfactory a still greater degree of accuracy may be attained, but at the cost of added labor of computation.

Referring to the tabulated statement on page 116, and considering I' a variable value of I, it will be seen that the following two sums of terms containing I' enter Eq. 36:

$$\int_0^l \frac{M_b\,y\,dx}{I'} - \frac{M''}{l}\int_0^l \frac{y\,x\,dx}{I'} - \frac{M'}{l}\int_0^l \frac{(l-x)\,y\,dx}{I'} + \frac{8f}{l^3}\Big((B+$$

$$C)\int_0^l \frac{M_b\,x\,dx}{I'} - Cl\int_0^l \frac{M_b\,dx}{I'}\Big) + \frac{8fk}{l^2}\int_0^l \frac{[(B+C)\frac{x}{l} - C]^2}{I'}$$

$$dx + \frac{w_0}{2}\left\{\int_0^l \frac{y\,x\,(l-x)\,dx}{I'} + \frac{8f}{l^3}(B+C)\int_0^l \frac{x^2(l-x)\,dx}{I'} - \right.$$

$$\left.\frac{8f}{l^2}C\int_0^l \frac{x\,(l-x)\,dx}{I'}\right\} + \int_{00}^{l_1 l_1} \frac{M_b\,y_1\,dx}{I'_1} - \frac{M'}{l_1}\int_{00}^{l_1 l_1} \frac{y_1\,x\,dx}{I'_1}$$

$$- \frac{8f_1}{l_1^3}C\int_{00}^{l_1 l_1} \frac{M_b\,x\,dx}{I'_1} + \frac{8f_1 k}{l_1^4}C^2\int_{00}^{l_1 l_1} \frac{x_2\,dx}{I'_1} + \frac{w_0}{2}\left\{\int_{00}^{l_1 l_1}\right.$$

$$\frac{y_1 x (l_1 - x)\, dx}{I'_1} - \frac{8 f_1}{l_1^3} C \int_{0\,0}^{l_1 l_1} \frac{x^2 (l_1 - x)\, dx}{I'_1} \Big\} \quad . \quad . \quad . \quad . \quad 39.$$

$$\int_0^l \frac{y^2\, dx}{I'} + \frac{8 f}{l^3} \left((B + C) \int_0^l \frac{y x\, dx}{I'} - C l \int_0^l \frac{y\, dx}{I'} \right) + \frac{64 f^2}{l^4}$$

$$\int_0^l \frac{[(B + C)\frac{x}{l} - C]^2\, dx}{I'} + \int_{0\,0}^{l_1 l_1} \frac{y_1^2\, dx}{I'_1} - \frac{8 f_1}{l_1^3} C \int_{0\,0}^{l_1 l_1} \frac{x\, y_1\, dx}{I'_1} +$$

$$\frac{64\, f_1^2\, C^2}{l_1^6} \int_{0\,0}^{l_1 l_1} \frac{x^2\, dx}{I'_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 40.$$

If the spans l_1 and l_3 are not of the same length the integrals with double limits must each be displaced by two integrals, one for span l_1 and the other for span l_3, as already explained.

The moments of inertia I' and I'_1 in the expressions 39 and 40 have constant values over parts of the spans l and l_1, as determined in the design of the trusses. When the limits of those parts are thus fixed the integrations in 39 and 40 can easily be made for the same limits, as I' and I'_1 are each constant for each part, although they vary from one part to another. The moments M_b are known quantities, as will presently be shown.

In general, let the parts of a span over which the moment of inertia I' or I'_1 is constant be limited by the horizontal coordinates x_1 and x, the length of each such part being $x - x_1$. Then in the actual use of the sums of terms 39 and 40 there will be needed the following definite integrals:

$$\int_{x_1}^{x} y x (l - x)\, dx = \frac{f}{l} l x^3 \left(\frac{4}{3} \left(1 - \frac{x_1^3}{x^3}\right) - \frac{2 x}{l} \left(1 - \frac{x_1^4}{x^4}\right) + \frac{4 x^2}{5 l^2} \left(1 - \frac{x_1^5}{x^5}\right) \right) 41.$$

$$\int_{x_1}^{x} y (l - x)\, dx = \frac{f}{l} l x^2 \left(2 \left(1 - \frac{x_1^2}{x^2}\right) - \frac{8 x}{3 l} \left(1 - \frac{x_1^3}{x^3}\right) + \frac{x^2}{l^2} \left(1 - \frac{x_1^4}{x^4}\right) \right) \quad 42.$$

$$\int_{x_1}^{x} y x\, dx = \frac{f}{l} l x^2 \left(\frac{4 x}{3 l} \left(1 - \frac{x_1^3}{x^3}\right) - \frac{x^2}{l^2} \left(1 - \frac{x_1^4}{x^4}\right) \right) \quad . \quad . \quad . \quad . \quad 43.$$

$$\int_{x_1}^{x} y^2\, dx = 4 \frac{f^2}{l^2} x^3 \left(\frac{4}{3} \left(1 - \frac{x_1^3}{x^3}\right) - \frac{2 x}{l} \left(1 - \frac{x_1^4}{x^4}\right) + \frac{4 x^2}{5 l^2} \left(1 - \frac{x_1^5}{x^5}\right) \right) \quad 44.$$

$$\int_{x_1}^{x} y\, dx = 2 \frac{f}{l} l^2 \left(\frac{x^2}{l^2} - \frac{x_1^2}{l^2} - \frac{2}{3} \left(\frac{x^3}{l^3} - \frac{x_1^3}{l^3} \right) \right) \quad . \quad . \quad . \quad . \quad 45.$$

$$\int_{x_1}^{x} x\,(l-x)\,dx = \frac{l^2}{2}\left(\frac{x^2}{l^2} - \frac{x_1^2}{l^2} - \frac{2}{3}\left(\frac{x^3}{l^3} - \frac{x_1^3}{l^3}\right)\right) \quad . \quad . \quad . \quad 46.$$

$$\int_{x_1}^{x} x^2\,(l-x)\,dx = \frac{l^4}{4}\left(\frac{4}{3}\left(\frac{x^3}{l^3} - \frac{x_1^3}{l^3}\right) - \left(\frac{x^4}{l^4} - \frac{x_1^4}{l^4}\right)\right) \quad . \quad 47.$$

$$\int_{x_1}^{x} (l-x)\,dx = \frac{l^2}{2}\left(2\left(\frac{x}{l} - \frac{x_1}{l}\right) - \left(\frac{x^2}{l^2} - \frac{x_1^2}{l^2}\right)\right) \quad . \quad . \quad . \quad 48.$$

$$\int_{x_1}^{x} x^2\,dx = \frac{l^3}{3}\left(\frac{x^3}{l^3} - \frac{x_1^3}{l^3}\right); \text{ and } \int_{x_1}^{x} x\,dx = \frac{l^2}{2}\left(\frac{x^2}{l^2} - \frac{x_1^2}{l_2}\right) . \quad 48a.$$

After determining the values of M_b, M'', and M' for any given case, it will be found that the preceding definite integrals are all which will be required for the use of expressions 39 and 40.

If it should be desired to determine the mean value I and the ratio $r = \frac{I'_1}{I}$ by the aid of expressions 39 and 40, that procedure may readily be followed. The sum of terms 39, determined as already explained, should then be placed equal to the corresponding sum expressed in terms of the mean value of I and of r in the numerator of the second member of Eq. 36. Similarly the sum of terms 40, also found as already explained, should be placed equal to the corresponding sum in the denominator of the second member of Eq. 36 expressed in terms of I and r. These two equations will at once give the desired values of I and r.

The preceding rather laborious processes will not often be required.

Three Cases of Stiffening-Trusses.

Three cases of stiffening-trusses may arise in connection with the general problem under consideration:

I. The stiffening-trusses may be continuous over all three spans with suspenders throughout those spans. Indeed they might be continuous over two of those spans and non-continuous over the other span, but that is an extremely unusual case, although Eq. 36 covers it.

II. The stiffening-trusses may be continuous over all three

spans with no suspenders in the side spans. In this case the extensions of the stiffening-trusses into the side spans may be much less in length than the full side spans or openings.

III. The stiffening-trusses may be non-continuous and they may be in the main span only, or in both the main and side spans.

Art. 44.—Case I. Continuous Stiffening-Trusses.

The value of H for this case is given by Eq. 36 precisely as it stands. It is only necessary to determine the integrals:

$$\int_0^l M_b\, y dx, \int_{0\,0}^{l_1\,l_1} M_b y_1 dx, \int_0^l M_b\, x dx, \int_{0\,0}^{l_1\,l_1} M_b x dx \text{ and } \int_0^l M_b dx. \quad 49.$$

for a stiffening-truss continuous throughout all three spans, remembering that if the side spans are of unequal length there will be two integrals in place of the second of 49, one for span l_1 and one for span l_3; and similarly for other like terms in Eq. 36.

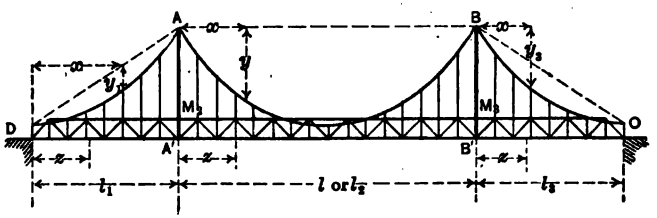

FIG. 13.

The values of M_b (the moving load bending moment at any point of any span considered an ordinarily supported continuous beam) must be found by the aid of the theorem of three moments. The dead load or own weight of the trusses, it should be observed, is carried at mean temperature directly by the suspenders to the cables, and thus fails to cause any bending whatever in the trusses at that temperature.

Considering the stiffening-trusses as ordinary continuous beams, it is clear, in Fig. 13, that the bending moments at D and O are zero. The bending moments M_2 and M_3 at the feet of

the towers A' and B', with any loads P on any or all spans, will be as shown on page 98:

$$M_3 = l_3 \frac{a^2 \overset{1}{\Sigma} P(1-n^2)n - (2a+r) \overset{2}{\Sigma} P(1-q^2)q - 2b^2(a+r)r^1 \overset{3}{\Sigma} P(1-m^2)m}{4(a+r)(br^1+1)-r} \quad . \quad 50.$$

$$M_2 = -l_2 \frac{r\frac{M_3}{l_3} + a^2 \overset{1}{\Sigma} P(1-n^2)n + r \overset{2}{\Sigma} P(1-q^2)q}{2(a+r)} \quad . \quad 51.$$

The notation in these two equations makes them perfectly general, as it assumes three different spans, l_1, l_2, and l_3, and three corresponding moments of inertia I_1, I (for span l_2), and I_3; it is the same as used elsewhere (writing I for I_2), viz.:

$$a = \frac{l_1}{l_2};\ b = \frac{l_3}{l_2};\ n = \frac{z}{l_1};\ m = \frac{z}{l_3};\ q = \frac{z}{l_2};\ I_1 = r\,I;\ \text{and}\ I = r'\,I_3.$$

It is also to be remembered that z, locating any load P, is measured from the left end of each span l_1, l_2, and l_3.

If M_0 represent the moving load bending moment in any span, considering the truss as simply supported and non-continuous as well as unconnected with the suspenders, the value of M_b for any span may at once be written by the aid of Eqs. 50 and 51 as follows:*

For span l_1 . . . $M_b = M_2 \frac{x}{l_1} + M_0$ . . . . . 52.

For span l_2 . . . $M_b = M_2 \frac{l_2 - x}{l_2} + M_3 \frac{x}{l_2} + M_0$ . . . 53.

For span l_3 . . . $M_b = M_3 \frac{l_3 - x}{l_3} + M_0$ . . . . . 54.

In these equations M_0 has the value $P\left(\frac{l-z}{l}\right)x$ between the ordinates o and z, and the value $P\frac{z}{l}(l-x)$ for the ordinates x between l and z.

* See "Elasticity and Resistance of Materials of Engineering," by Wm. H. Burr, page 84.

The following integrals will then be needed below:

$$\int_0^l M_0(l-x)\,x\,dx = \frac{Pz}{l}\left(\frac{l^4}{12}-\frac{l^2z^2}{6}+\frac{lz^3}{12}\right) = Pz\frac{l^3}{12}\left(1-q^2(2-q)\right)$$

$$= \frac{pl^3}{60} \text{ (for uniform load } p);$$

$$\int_0^l M_0 x\,dx = \frac{Pz}{6}(l^2-z^2) = Pz\frac{l^2}{6}(1-q^2) = \frac{pl^4}{24} \text{ (for uniform load } p);$$

$$\int_0^l M_0\,dx = \frac{Pz}{2}(l-z) = Pz\frac{l}{2}(1-q) = \frac{pl^3}{12} \text{ (for uniform load } p).$$

Taking the values of y and y_1 from Eq. 32, and remembering that x may be measured from either end of each span in Eqs. 55, there may now be written:

$$\left.\begin{aligned}
\int_0^{l_2} M_b y\,dx &= \frac{fl_2}{3}\left[M_2+M_3+\overset{s}{\Sigma}\,Pz\left(1-q^2(2-q)\right)\right]\\
\int_0^{l_1} M_b y_1\,dx &= \frac{f_1 l_1}{3}\left[M_2+\overset{1}{\Sigma}\,Pz\left(1-q^2(2-q)\right)\right]\\
\int_0^{l_3} M_b y_3\,dx &= \frac{f_3 l_3}{3}\left[M_3+\overset{s}{\Sigma}\,Pz\left(1-q^2(2-q)\right)\right]
\end{aligned}\right\} \quad . \quad . \quad . \quad 55.$$

$$\left.\begin{aligned}
\int_0^{l_1} M_b x\,dx &= M_2\frac{l_1^2}{3}+\overset{1}{\Sigma}\,Pz\,\frac{l_1^2}{6}(1-q^2)\\
\int_0^{l_2} M_b x\,dx &= (M_2+2M_3)\frac{l_2^2}{6}+\overset{s}{\Sigma}\,Pz\frac{l_2^2}{6}(1-q^2)\\
\int_0^{l_3} M_b x\,dx &= M_3\frac{l_3^2}{6}+\overset{s}{\Sigma}\,Pz\,\frac{l_3^2}{6}(1-q^2)
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad 56.$$

$$\int_0^{l_2} M_b\,dx = (M_2+M_3)\frac{l_2}{2}+\overset{s}{\Sigma}\,Pz\frac{l_2}{2}(1-q) \quad . \quad . \quad . \quad . \quad . \quad 57.$$

Usually P in these equations will be the panel moving load applied at the panel points, and z will be the distance of the panel point at which P is applied from one end of the span. The distance z therefore may be expressed in panel lengths.

The summations Σ, in Eqs. 55, 56, and 57, may be quickly made by the aid of the following table:

q	$(1-q^2)$	$1-q^2(2-q)$	q	$(1-q^2)$	$1-q^2(2-q)$
.05	.9975	.995125	.55	.6975	.561375
.1	.99	.981	.6	.64	.496
.15	.9775	.958375	.65	.5775	.429625
.2	.96	.928	.7	.51	.363
.25	.9375	.890625	.75	.4375	.296875
.3	.91	.847	.8	.36	.232
.35	.8775	.797875	.85	.2975	.169125
.4	.84	.744	.9	.19	.109
.45	.7975	.686125	.95	.0975	.052375
.5	.75	.625	1.	.00	.00

Greater convenience, even, may be attained by constructing a curve with q as the abscissa and $1 - q^2 (2 - q)$ as the ordinate.

Uniform Load.

The last term of the second members of Eqs. 55, 56, and 57 can easily be expressed for uniform load by writing $p\,dz$ for P, p being the intensity of load or the uniform load per linear unit (foot). The expression then becomes, the total uniform load being $p\,(x - x_1)$:

$$\sum_{x_1}^{x} Pz\left(1-q^2\,(2-q)\right) = \frac{1}{l^3}\int_{x_1}^{x} pz\left(l^3-z^2(2\,l-z)\right)dz$$

$$= \frac{p\,l^2}{2}\left[\left(\frac{x^2}{l^2}-\frac{x_1^2}{l^2}\right)-\left(\frac{x^4}{l^4}-\frac{x_1^4}{l^4}\right)+\frac{2}{5}\left(\frac{x^5}{l^5}-\frac{x_1^5}{l^5}\right)\right] . \quad 58.$$

Eq. 58 covers the most general case of a given length $(x - x_1)$ of uniform load on any part of any span, as indicated for the main span in Fig. 13.

If the uniform load extends from one end of the span over the distance x, $x_1 = 0$, and Eq. 58 becomes:

$$\sum_{0}^{x} Pz\left(1-q^2\,(2-q)\right) = \frac{p\,l^2}{2}\left(\frac{x^2}{l^2}-\frac{x^4}{l^4}+\frac{2}{5}\frac{x^5}{l^5}\right) . \quad 59.$$

Finally if the load covers the entire span, $x = l$, and Eq. 58 or 59 gives:

$$\sum_0^l Pz\left(1 - q^2\,(2 - q)\right) = \frac{p\,l^2}{5} \quad . \quad . \quad . \quad 60.$$

Similarly, the last term of the second member of Eq. 56 becomes for the uniform loading $p\,(x - x_1)$:

$$\frac{Pz}{6}(l^2 - z^2)\,dz = \frac{p}{6}\int_{x_1}^{x} z(l^2 - z^2)\,dz = \frac{p l^4}{24}\left(2\left(\frac{x^2}{l^2} - \frac{x_1^2}{l^2}\right) - \left(\frac{x^4}{l^4} - \frac{x_1^4}{l^4}\right)\right) \quad .61.$$

If the loading p covers the distance x from one end of the span, $x_1 = 0$ and:

$$\frac{p}{6}\int_0^x z\,(l^2 - z^2)\,dz = \frac{p\,l^4}{24}\left(\frac{2x^2}{l^2} - \frac{x^4}{l^4}\right). \quad . \quad . \quad . \quad 62.$$

If the load p covers the entire span, $x = l$ and:

$$\frac{p}{6}\int_0^l z\,(l^2 - z^2)\,dz = \frac{p\,l^4}{24} \quad . \quad . \quad . \quad . \quad . \quad 63.$$

Finally, treating in the same manner the last term of the second member of Eq. 57 for the uniform loading $p\,(x - x_1)$:

$$\frac{Pz}{2}\,(l - z) = \frac{p}{2}\int_{x_1}^{x} z(l - z)\,dz = \frac{p l^3}{12}\left(3\left(\frac{x^2}{l^2} - \frac{x_1^2}{l^2}\right) - 2\left(\frac{x^3}{l^3} - \frac{x_1^3}{l^3}\right)\right) \quad 64.$$

If the loading extends from one end of the span over the distance x, $x_1 = 0$ and:

$$\frac{p}{2}\int_0^x z\,(l - z)\,dz = \frac{p\,l^3}{12}\left(3\,\frac{x^2}{l^2} - 2\,\frac{x^3}{l^3}\right) \quad . \quad . \quad 65.$$

When the entire span is covered by the load, $x = l$ and:

$$\frac{p}{2}\int_0^l z\,(l - z)\,dz = \frac{p\,l^3}{12} \quad . \quad . \quad . \quad . \quad . \quad 66.$$

Eqs. 58 to 66 are made applicable to spans l_1, l_2, and l_3 by simply using the proper subscript 1, 2, or 3.

Eqs. 55, 56, and 57 with their adaptations to the special cases of uniform load now enable Eq. 36 to be used in any case of continuous trusses whatever. The areas of cross-section F_0, F_2, F_3, and F_1 must be used as tentative values found by some

simple approximate methods, as those of Chapter I, to be revised if necessary after computing the value of H.

Art. 45.—Case II. Continuous Stiffening-Trusses, but No Suspenders in Side Spans.

In this case the cables in the side spans become anchor cables or backstays, practically straight, requiring the use of the item "Anchor cable or backstay" in the tabulation preceding Eq. 36 instead of the item "Side cable." This necessitates substituting $2\,l'_1\,sec^2\,\alpha_1$ for $2\,l_1\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)$ in the denominator of the second member of Eq. 36.

Again, as there are no suspenders in the side spans, the terms $\frac{16\,f_1^2}{l_1}\left(\frac{1}{2}\frac{h''}{f_1}-\frac{2}{3}\right)$, in the numerator, and $\frac{128\,f_1^3}{l_1^3}\left(\frac{1}{2}\frac{h''}{f_1}-\frac{2}{3}\right)$ in the denominator, of the second member of Eq. 36 must be omitted.

Finally, the resultant bending moment in the side spans will not be affected by the suspender pull, except as that pull affects the end moments M', since the stiffening-truss of length l_1 or l_3 will be traversed by the moving load and carry its own weight without being attached to suspenders. This necessitates omitting the term $\frac{F_0}{r\,I}\frac{16}{15}f_1^2\,l_1$, in the denominator, of the second member of Eq. 36.

Furthermore, as y_1 is now zero, the resultant bending moment M in a side span will now be:

$$M = M_b + \frac{w_0\,x}{2}(l_1 - x) - M'\frac{x}{l_1};\ \text{and}\ \frac{d\,M}{d\,H} = \frac{8f}{l^2}C\frac{x}{l_1}.$$

Therefore, in the tabulated statement preceding Eq. 36 and under the heading $E\frac{d\,x}{d\,H}$ for the side spans, there should be retained for this case only the terms:

$$+\frac{8f}{l_1 l^2}C\int_{00}^{l_1 l_1} M_b\,x\,d\,x + \left(H\frac{8f}{l^2}-k\right)\frac{8f}{l^2 l_1^2}C^2\int_{00}^{l_1 l_1} x^2\,d\,x$$

$$+\frac{w_0}{2}\frac{8f}{l^2 l_1}C\int_{00}^{l_1 l_1} x^2(l_1 - x)\,dx.$$

All the remaining terms due to the bending moments in the side spans contain y_1 which is now zero.

Again, as the upward pull of the suspenders is exerted in the centre span only, by referring to Eqs. 19 and 20 of Art. 42:

$$M'' = -\frac{s l_2^2}{4}\frac{2a+r}{4(a+r)(br'+1)-r} = -sB \quad . \quad 67.$$

$$M' = -\frac{s l_2^2 r}{8(a+r)}\frac{2(a+r)(br'+1)}{4(a+r)(br'+1)-r} = -sC \quad . \quad 68.$$

The values of B and C in Eq. 69 must be taken from Eqs. 67 and 68.

Making all the changes above indicated in Eq. 36:

$$H = \frac{k\frac{F_0}{F_2}\frac{8f^2}{l}\left(\frac{h''}{f}-\frac{2}{3}\right) + \frac{F_0}{I}\left\{\int_0^l M_b y\,dx + \frac{w_0}{15}\frac{f}{l}l^4 + \frac{8f}{l^3}\right\}(B+C)}{l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)+2l_1\sec^2\alpha_1+\frac{F_0}{F_2}\frac{64f^3}{l^3}\left(\frac{h''}{f}-\frac{2}{3}\right)+\frac{F_0}{F_3}2h'(\tan\alpha+\tan\alpha_1)^2}$$

$$\frac{\int_0^l M_b x\,dx - Cl\int_0^l M_b\,dx\Big\} - \frac{8f}{rl_1l^2}C\int_{00}^{l_1 l_1} M_b\,x\,dx\Big\} + \frac{F_0}{3}\frac{k}{I}\Big\{(B-C)fl}$$

$$+\frac{F_0}{I}\left(\frac{8}{15}f^2 l + \frac{16f^2}{3l}(B-C) + \frac{64f^2}{3l^3}\left(B^2 - C(B-C)\right)\right.$$

$$\left.+\frac{8f}{l}\left(B^2 - C(B-C)\right) + \frac{16fl_1}{rl^2}C^2\right\} + \frac{F_0 w_0}{I}\left(\frac{fl}{3}(B-C) - \frac{2f}{3rl^2}l_1^3 C\right)$$

$$+\frac{128f^2 l_1}{3l^4 r}C^2\Big) \quad . \quad 69.$$

In this equation, l'_1 is the total length of the side span from the centre of the tower to the anchorage, so that $l'_1 \sec\alpha_1$ is the length of the backstay.

The first term of Eq. 69 becomes H' if the terms containing w_0 or k are made zero in the numerator of the second member, remembering that H'' may be computed by taking the dead load as suspended from the cable.

Values of $\int_0^l M_b\, y\, dx, \int_0^l M_b\, x\, dx$, *etc.*

The determination of these integrals will depend upon considerations set forth at length in Art. 63 regarding special features of this case.

Under these conditions the dead load moments will not enter the values of M_b for any span, as already observed.

It should be carefully discriminated that the dead load of the centre or main span is carried directly by the cable through the suspenders while that of the side spans rests upon those

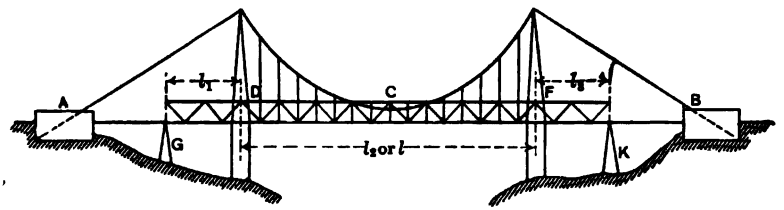

FIG. 14.

spans as simple non-continuous trusses at mean temperature. The effects of temperatures other than the mean will be considered as an independent matter.

Inasmuch as any moving load on any of the three spans, l_1, l_2, and l_3, affects the value of H (or, since H is affected by the moving load bending moment in any of those three spans), the amount of work to be made a minimum in finding H includes the work performed by the moving load moments or stresses in all those spans. Hence the values of M_b are the same as those given by Eqs. 52, 53, and 54 in the preceding article. For the same reason the values of $\int_0^l M_b\, y\, dx, \int_0^l M_b\, x\, dx, \int_{00}^{l_1 l_1} M_b\, x\, dx$, and $\int_0^l M_b\, dx$ are given by Eqs. 55, 56, and 57 of the same article.

With the complete determination of these values, Eq. 69 may at once be applied to find the horizontal component, H, of the cable stress.

The dead load stresses in the spans l_1 and l_3 are to be found

as if the trusses of those spans were non-continuous and simply supported at the ends, then tabulated and combined with the moving load stresses in the same spans.

The spans l_1 and l_2 are extensions of the stiffening-trusses into the side spans, as shown in Fig. 14, and their ends may require anchorage.

Art. 46.—Case III. Non-Continuous Stiffening-Trusses—Stiffening-Trusses in One or Three Spans.

Two conditions may arise when the stiffening-trusses are non-continuous; the latter may be found in all three spans, or there may be stiffening-trusses in the centre or main span only. The preceding equations, with simple modifications, cover either condition.

Stiffening-Trusses in All Three Spans.

Eq. 36 gives the value of H for this case precisely as it stands after dropping all terms depending upon continuity of the trusses. It is only necessary to observe that M_b for any span or all spans is the moving load bending moment in a non-continuous, simply supported truss loaded in any manner. The dead load is carried directly to the cables by the suspenders and produces no bending in the stiffening-trusses at mean temperature.

Every term in Eq. 36, Art. 43, containing B or C must be dropped, as all such terms belong only to continuous stiffening-trusses. These changes will give:

$$H = \frac{\dfrac{F_0}{F_2}\,k\left\{\dfrac{8f^2}{l}\left(\dfrac{h''}{f}-\dfrac{2}{3}\right)+\dfrac{16f_1^2}{l_1}\left(\dfrac{1}{2}\dfrac{h''}{f_1}-\dfrac{2}{3}\right)\right\}}{l\left(1+\dfrac{16}{3}\dfrac{f^2}{l^2}\right)+2l_1\left(1+\dfrac{16}{3}\dfrac{f^2}{l^2}\right)+\dfrac{F_0}{F_2}\left\{\dfrac{64f^2}{l^2}\left(\dfrac{h''}{f}-\dfrac{2}{3}\right)\right.}$$

$$+\frac{F_0}{I}\left\{\int_0^l M_b\,y\,dx+\frac{w_0}{15}\frac{f}{l}\,l^4+\frac{1}{r}\left(\int_0^{l_1}\int_0^{l_1} M_b\,y_1\,dx\right.\right.$$

$$\left.\left.+\frac{128f_1^2}{l_1^2}\left(\frac{1}{2}\frac{h''}{f_1}-\frac{2}{3}\right)\right\}+\frac{F_0}{F_2}\,2h'\,(\tan\alpha+\tan\alpha_1)^2\right.$$

$$+\frac{2\,w_0 f_1}{15\,l_1} l_1^4\Big)\Big\}$$

$$+\frac{F_0}{I}\left(\frac{8}{15} f^2 l + \frac{16}{15\,r} f_1^2 l_1\right) \quad . \quad . \quad . \quad . \quad 70.$$

Eq. 70 gives the general value of H for non-continuous stiffening-trusses with suspenders in all three spans.

If the spans l_1 and l_3 are not of equal length, the paragraph following Eq. 37 must be carefully regarded.

Eqs. 55 of Art. 44 show for this case:

$$\int_0^{l_2} M_b\, y\, d\,x = \frac{f\,l_2}{3}\left[\Sigma^{\mathrm{s}}\, P\, z\,(1 - q^2\,(2 - q))\right] \quad . \quad . \quad 71.$$

$$\int_0^{l_1} M_b\, y_1\, d\,x = \frac{f_1\,l_1}{3}\left[\Sigma^{\mathrm{1}}\, P\, z\,(1 - q^2\,(2 - q))\right] \quad . \quad . \quad 72.$$

$$\int_0^{l_3} M_b\, y_3\, d\,x = \frac{f_3\,l_3}{3}\left[\Sigma^{\mathrm{s}}\, P\, z\,(1 - q^2\,(2 - q))\right] \quad . \quad . \quad 73.$$

Eqs. 56 and 57 are not needed when the trusses are non-continuous.

In these equations, in general, $q = \frac{z}{l}$.

Usually, as already stated, P will be the panel moving loads applied at the panel points, at distances z from one end of the span.

As with Eqs. 69 and 36, Eq. 70 will give H' when all terms containing w_0 or k in the numerator of the second member are dropped.

If the load is uniform, Eqs. 58, 59, and 60 show the values taken by Eqs. 71, 72, and 73 for any length of uniform load placed anywhere in any span. The tables of numerical values given in connection with Eqs. 55 to 66 of Art. 44 greatly shorten actual computations.

Stiffening-Trusses in Centre or Main Span Only.

In this case the side cables become anchor cables or back-stays, as there are neither suspenders nor stiffening-trusses in the side spans. Omitting the corresponding terms in Eq. 70, the

value of H, the horizontal component of cable stress, is then at once given by Eq. 74:

$$H = \frac{\frac{F_0}{F_2}\frac{8f^2}{l}k\left(\frac{h''}{f}-\frac{2}{3}\right)+\frac{F_0}{I}\left\{\int_0^l M_b\, y\, dx + \frac{w_0 f}{15 l} l^4\right\}}{l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)+2l_1\sec^2\alpha_1+\frac{F_0}{F_2}\frac{64 f^3}{l^3}\left(\frac{h''}{f}-\frac{2}{3}\right)+\frac{F_0}{F_3}2h'(\tan\alpha+\tan\alpha_1)^2+\frac{F_0}{I}\frac{8}{15}f^2 l} \quad . \quad 74.$$

If the two side spans are not of the same length ($l_1 \sec^2\alpha_1 + l_3 \sec^2\alpha_3$) should be written for $2\, l_1 \sec^2\alpha_1$.

The value of $\int_0^l M_b\, y\, dx$ is to be taken from Eq. 71. In case of uniform load, resort again should be made to Eqs. 58, 59, and 60 of Art. 44.

If both numerator and denominator of the second member of Eq. 74 be divided by the third term of the denominator:

$$H = \frac{\frac{k l^2}{8f}+\frac{F_2}{I}\frac{1}{\frac{64 f^3}{l^3}\left(\frac{h''}{f}-\frac{2}{3}\right)}\left(\int_0^l M_b\, y\, dx + \frac{w_0 f}{15 l} l^4\right)}{1+\frac{1}{\frac{64 f^3}{l^3}\left(\frac{h''}{f}-\frac{2}{3}\right)}\left\{\frac{F_2}{F_0}l\left(1+\frac{16 f^2}{3 l^2}+2\frac{l_1}{l}\sec^2\alpha_1\right)+\frac{F_2}{F_3}2h'(\tan\alpha+\tan\alpha_1)^2+\frac{F_2}{I}\frac{8}{15}\right) f^2 l} \quad . \quad 75.$$

If in the second member of Eq. 75 all terms depending upon the work expended in stressing the different parts of the structure, and, hence upon the loads causing the stresses, be omitted, there will result:

$$H = \frac{k l^2}{8f} \quad . \quad . \quad . \quad . \quad . \quad . \quad 76.$$

Eq. 75, therefore, shows the influence of the elastic deformations or strains on the value of H, Eq. 76 giving H for the cable weight only, when all such deformations or strains are disregarded.

Art. 47.—Bending Moments in Continuous Stiffening-Trusses.

Having determined the value of the horizontal component, H, of the cable stress for the various given conditions, all the stresses in both the cable and the stiffening-trusses may at once be completely computed. To do this the moving load bending moments and shears must first be found.

CASE I.—CONTINUOUS STIFFENING-TRUSSES.

The value of H for this case, together with other formulæ, will be found set forth in Art. 44.

Bending Moments in Stiffening-Trusses.

The general value of the bending moments for the three spans l_1, l_2, and l_3 are given by Eqs. 8*b*, 8*c*, and 8*d* of Art. 42.

By Eq. 32 of Art. 43:

$$y_2 = \frac{4 f_2}{l_2^2} (l_2 - x)\, x; \quad \ldots \quad 77.$$

$$y_1 = \frac{4 f_1}{l_1^2} (l_1 - x)\, x; \quad \ldots \quad 78.$$

$$y_3 = \frac{4 f_3}{l_3^2} (l_3 - x)\, x. \quad \ldots \quad 79.$$

The moments M' and M'' at the points of support caused by the uniform upward moving load pull of the suspenders $s' = H^1 \frac{8 f_2}{l_2^2}$, as found by Eq. 33*a* of Art. 43, are given as follows by Eqs. 19 and 20 of Art. 42:

$$M'' = \frac{s' l_2^2}{4} \frac{a^3 - (2a + r) - 2 b^3 (a + r) r'}{4 (a + r)(b r' + 1) - r} = s' B; \quad 80.$$

$$M' = -\frac{s' l_2^2}{8 (a+r)} \left\{ \frac{r(a^3 - (2a+r) - 2 b^3 (a+r) r')}{4 (a+r)(b r^1 + 1) - r} + a^3 + r \right\} = -s^1 C. \quad 81.$$

The moments M_b, caused by moving load only, are given by Eqs. 52, 53, and 54 of Art. 44, M_2 and M_3 being given by Eqs. 50 and 51 of the same Article.

Introducing the values of y and M_b in Eqs. 8*b*, 8*c*, and 8*d*: For span l_1:

$$M = M_2 \frac{x}{l_1} + M_0 - H'\left(\frac{4 f_1}{l_1^2}(l_1 - x)\, x\right) - M' \frac{x}{l_1} \quad . \quad . \quad 82.$$

For span l_2:

$$M = M_2 \frac{l_2 - x}{l_2} + M_3 \frac{x}{l_2} + M_0 - H'\left(\frac{4 f_2}{l_2^2}(l_2 - x)x\right) - M'' \frac{x}{l_2} - M' \frac{l_2 - x}{l_2} \quad . \quad . \quad . \quad . \quad 83.$$

For span l_3:

$$M = M_3 \frac{l_3 - x}{l_3} + M_0 - H'\left(\frac{4 f_3}{l_3^2}(l_3 - x)x\right) - M'' \frac{l_3 - x}{l^3} \quad . \quad 84.$$

These equations are perfectly general for any amounts or positions of loading on any or all spans. They may be somewhat simplified for the usual conditions of actual construction.

The value of the moving load moments M_3 and M_2 over the points of support at the extremities of the main span l_2 are given by Eqs. 14 and 15 of Art. 42, or Eqs. 50 and 51 of Art. 44 for concentrated loads, but uniform loads may be used in nearly all cases. Let it first be assumed that the uniform loads per linear foot for the span l_1, l_2, and l_3 are p_1, p_2, and p_3, respectively; then let the length of that uniform load in each span be $z - z_1$, z and z_1 being measured from the left end of each span, *i.e.*, from the same origin as x.

In Eqs. 50 and 51 of Art. 44, $p\, d\, z$ is to be written for P, and the typical operation of changing from summation to integration will then be:

$$\sum_{z_1}^{z} P\,(1 - n^2)\, n = \frac{p}{l^3}\int_{z_1}^{z} (l^2 - z^2)\, z\, d\, z = \frac{p}{4\, l^3}\left(2\, l^2(z^2 - z_1^2) - (z^4 - z_1^4)\right)$$

$$= \frac{p\, l}{4}\left((2\, n^2 - n^4) - (2\, n_1^2 - n_1^4)\right) \quad . \quad . \quad . \quad . \quad 85.$$

There will result by the use of this typical equation:

$$M_3 = \frac{l_2^2}{4} \frac{p_1 a^3 ((2n^2-n^4) - (2n_1^2-n_1^4)) - (2a+r) p_2 ((2q^2-q^4)}{4(a+r)(br'+1) - r}$$
$$\underline{-(2q_1^2-q_1^4)) - 2b^3(a+r)r'p_3((2m^2-m^4)-(2m_1^2-m_1^4))} \quad . \quad 86.$$

$$M_2 = -\frac{l_2^2}{4} \frac{\frac{4rM_3}{l_2^2} + p_1 a^3 ((2n^2-n^4)-(2n_1^2-n_1^4)) + rp_2 ((2q^2-q^4)}{2(a+r)}$$
$$\underline{-(2q_1^2-q_1^4))} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 87.$$

The uniform moving load would usually be the same on all spans, or, $p_1 = p_2 = p_3 = p$.

Again, generally, $l_1 = l_3$, or $a = b$. Furthermore, these spans may be half the length of the main span, *i.e.*, $a = b = \frac{1}{2}$. Also $I_1 = I_3$, or $r = \frac{1}{r'}$. If it should be permissible to take $I_1 = I_2 = I_3$, then $r = r' = 1$.

The smallest length of uniform load would be a panel length, and it can be taken as a single or concentrated panel load, although, strictly speaking, uniform moving load would have to cover at least two panels to secure a full panel concentration at a panel point.

The following table will save labor in the application of Eqs. 86 and 87.

n	$2n^2-n^4$	n	$2n^2-n^4$
.05	.00499375	.55	.51349375
.1	.0199	.6	.5904
.15	.04449375	.65	.66649375
.2	.0784	.7	.7399
.25	.12109375	.75	.80859375
.3	.1719	.8	.8704
.35	.22999375	.85	.91999375
.4	.2944	.9	.9639
.45	.36399375	.95	.99049375
.5	.4375	1.	1.

Although the table has the headings n and $2n^2 - n^4$, it obviously applies equally to the ratios q and m.

If, again, the end spans are equal, $l_1 = l_3$, so that $a = b$, and if, as usual, $r = \frac{I}{r^1}$, M' becomes equal to M''. When these substitutions are made in Eqs. 80 and 81 they become:

$$M' = M'' = -\frac{s' l_2^2}{4} \frac{a^2 + r}{2a + 3r} \quad . \quad . \quad . \quad . \quad 88.$$

In the second member of Eq. 83, therefore:

$$-M'' \frac{x}{l_2} - M' \frac{l_2 - x}{l_2} = -M' = -M'' \quad . \quad . \quad 89.$$

Making the substitutions in Eqs. 82, 83, and 84 indicated by Eqs. 88 and 89, there will result:

For span l_1:

$$M = M_2 \frac{x}{l_1} + M_0 - H' \left(\frac{4 f_1}{l_1^2} (l_1 - x) x \right) + \frac{s' l_2^2}{4} \frac{a^2 + r}{2a + 3r} \frac{x}{l_1} \quad . \quad 91.$$

For span l_2:

$$M = M_2 \frac{l_2 - x}{l_2} + M_3 \frac{x}{l_2} + M_0 - H' \left(\frac{4 f_2}{l_2^2} (l_2 - x) x \right) + \frac{s' l_2^2}{4} \frac{a^2 + r}{2a + 3r} \quad . \quad . \quad . \quad . \quad . \quad 92.$$

For span l_3:

$$M = M_3 \frac{l_3 - x}{l_3} + M_0 - H' \left(\frac{4 f_3}{l_3^2} (l_3 - x) x \right) + \frac{s' l_2^2}{4} \frac{a^2 + r}{2a + 3r} \frac{l_3 - x}{l_3} \quad . \quad . \quad . \quad . \quad . \quad 93.$$

As $l_1 = l_3$ in these equations, it should be observed that the subscripts are retained to distinguish the spans and not to indicate different span lengths.

In these equations the suspender pull s' per linear unit of span is the uniform pull due to moving load only.

$$s' = H' \frac{8 f}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 94.$$

It will be observed that Eq. 91 may be used for the span l_3 by changing l_1 to l_3, M_2 to M_3, and f_1 to f_3, but x and z in l_3 must then be measured from the right-hand end of that span. The same result will be obtained by writing $l_3 - x$ for x in Eq. 93.

Art. 48.—Points of Contraflexure in Continuous Spans l_1 and l_3.

The points of contraflexure in a stiffening-truss are found precisely as in the case of any ordinary continuous beam, *i.e.*, by placing the bending, $M = 0$, and solving the equation for x. The general equations to be used for this purpose are Eqs. 82, 83, and 84. It will be sufficient, however, to illustrate this part of the analysis by the use of Eqs. 91, 92, and 93, in which $l_1 = l_3$ and $r = r'$, as may be the case. The uniform moving load may be considered a series of uniform panel loads, K,* applied at the panel points. The value of M_0 for that part of any span between its left end and the point of application of K (*i.e.*, for the part z of the span) will be for a single load:

$$M_o = K \frac{l - z}{l} x \quad . \quad . \quad . \quad . \quad . \quad 95.$$

For the part $(l - z)$ of the span:

$$M_0 = K \frac{z}{l} (l - x) \quad . \quad . \quad . \quad . \quad . \quad 96.$$

If the load is uniform and covers the entire span:

$$M_0 = \frac{p}{2} (l - x) x \quad . \quad . \quad . \quad . \quad . \quad . \quad 97.$$

If M_o, given by Eq. 95, be placed in Eq. 91, and if the second member of the latter be then placed equal to zero, x will be factored out, giving for the distance of the point of contraflexure from the left end A of the span, Fig. 13, Art. 44:

$$x = l_1 \left\{ 1 - \frac{1}{4 H' f_1} \left(M_2 + K(l_1 - z) + \frac{s' l_2^2}{4} \left(\frac{a^3 + r}{2a + 3r} \right) \right) \right\} \quad . \quad 98.$$

*Although this letter is used in Art. 43 to represent the uniform load per panel of the cable, no confusion will result from giving to it the meaning indicated here.

There is therefore one point of contraflexure between the left end of span l_1 and the point of application of the (panel) load K.

For the right-hand part $l_1 - z$ of the span l_1 the moment M_o must be taken from Eq. 96, and if that value be placed in Eq. 91, and if the second member of the latter equation be made equal to zero, there will result a simple equation of the second degree, the usual solution of which will give:

$$x = \frac{l_1}{8\,H' f_1}\left(4\,H' f_1 + Kz - M_2 - \frac{s' l_2^2}{4}\,\frac{a^2+r}{2\,a+3\,r}\right.$$
$$\left. \pm \sqrt{\left(4\,H' f_1 + Kz - M_2 - \frac{s' l_2^2}{4}\,\frac{a^2+r}{2\,a+3\,r}\right)^2 - 16\,Kz\,H' f_1}\right) . \quad 99.$$

Eq. 99 shows that there will be two points of contraflexure between the point of application of the load K and the right end of span l_1, making three such points in the span.

Again, if the span l_1 carries the uniform load p_1 per linear unit (per foot), the value of M_o is to be taken from Eq. 97 and placed in the second member of Eq. 91, which is then to be made equal to zero. This procedure will give:

$$x = l_1\left(1 - \frac{M_2 + \frac{s' l_2^2}{4}\,\frac{a^2+r}{2\,a+3\,r}}{4\,H' f_1 - \frac{1}{2}\,p_1 l_1^2}\right) \quad . \quad . \quad . \quad 100.$$

In this case there is then but one point of contraflexure in span l_1.

Eqs. 98, 99, and 100 apply directly to span l_2 as they stand by simply writing l_2 for l_1, f_2 for f_1, and M_3 for M_2, remembering that x and z will then be measured from the right-hand end of the span.

For the span l_2, carrying a single (panel) load K at distance z from its left end, M_o is to be taken from Eq. 95; and the second member of Eq. 92 then placed equal to zero will give for that part of the span between the left end and K:

$$x = \frac{l_2}{2}\left(1 - \frac{M_3 - M_2 + K(l_2 - z)}{4\,H' f_2} \pm \sqrt{\left(1 - \frac{M_3 - M_2 + k(l_2 - z)}{4\,H' f_2}\right)^2}\right.$$
$$\left. - \frac{M_2\,\frac{s' l_2^2}{4}\,\frac{3\,a^2+r}{2\,a+3\,r}}{H' f_2}\right) \quad . \quad . \quad . \quad . \quad 101.$$

This equation shows that there are two points of contraflexure between the load K and the left end of the span.

Eq. 96 gives the moment M_o for the part $l_2 - z$ of the span, and the substitution of it in Eq. 92, also making $M = 0$, will give:

$$x = \frac{l_2}{2}\left(1 - \frac{M_3 - M_2 - Kz}{4H'f_2} \pm \sqrt{\left(1 - \frac{M_3 - M_2 - Kz}{4H'f_2}\right)^2 - \frac{Kz + M_2 + \frac{s'l_2^2}{4}\frac{a^3 + r}{2a + 3r}}{H'f_2}}\right) \quad . \quad . \quad . \quad 102.$$

There are, therefore, two points of contraflexure in the part $l_2 - z$ of the beam, making four points in the span l_2, two on either side of the load K.

If the span l_2 is covered with a uniform moving load, p_2, per linear foot, Eq. 97 will give the value of M_o. By placing the second member of Eq. 92 equal to zero after substituting the value of M_o in it, there will result:

$$x = \frac{l_2}{2}\left(1 - \frac{M_3 - M_2}{4H'f_2 - \frac{1}{2}p_2 l_2^2} \pm \sqrt{\left(1 - \frac{M_3 - M_2}{4H'f_2 - \frac{1}{2}pl_2^2}\right) - \frac{M_2 + \frac{s'l_2^2}{4}\frac{a^3 + r}{2a + 3r}}{H'f_2 - \frac{1}{8}pl_2^2}}\right) \quad . \quad . \quad . \quad . \quad 103.$$

Eq. 103 shows that, with a uniform load on the entire span l_2, there will be but two points of contraflexure in the span.

These special instances of the points of contraflexure show sufficiently the procedures to be followed in all cases.

Art. 49.—Deflection of Continuous Stiffening-Trusses in Spans l_1 and l_2.

If u represent the deflection of the stiffening-truss at any point distant x from the left end of any span, there may be written by using Eqs. 91, 95, and 96:

$$EI\frac{d^2u}{dx^2} = M = M_2\frac{x}{l_1} + K\left(1 - \frac{z}{l_1}\right)_o^z x + Kz\left(1 - \frac{x}{l_1}\right)_z^{l_1} - H'4f_1\left(1 - \frac{x}{l_1}\right)\frac{x}{l_1} + \frac{s'l_2^2}{4}\frac{a^3 + r}{2a + 3r}\frac{x}{l_1} \quad . \quad . \quad 104.$$

Observing that $\int_o^x K\left(1-\frac{z}{l_1}\right)x\,dx = K\left(1-\frac{z}{l_1}\right)\frac{x^2}{2}$ and $\int_z^x Kz\left(1-\frac{x}{l_1}\right)dx = Kz(x-z)\left(1-\frac{x+z}{2l_1}\right)$, there will result:

$$EI\frac{du}{dx} = M_2\frac{x^2}{2l_1} + K\left(1-\frac{z}{l_1}\right)_o^z\frac{x^2}{2} + Kz(x-z)\left(1-\frac{x+z}{2l_1}\right)_z^{l_1} - H^1 4f_1\left(\frac{x^2}{2l_1}-\frac{x^3}{3l_1^2}\right) + \frac{s'l_2^2}{4}\frac{a^3+r}{2a+3r}\frac{x^2}{2l_1} + C \quad . \quad 105.$$

Again integrating,

$$EI_1 u = M_2\frac{x^3}{6l_1} + K\left(1-\frac{z}{l_1}\right)_o^z\frac{x^3}{6} + Kz\frac{(x-z)^2}{2}\left(1-\frac{2z+x}{3l_1}\right)_z^{l_1} - H^1 4f_1\left(\frac{x^3}{6l_1}-\frac{x^4}{12l_1^2}\right) + \frac{s'l_2^2}{4}\frac{a^3+r}{2a+3r}\frac{x^3}{6l_1} + Cx \quad . \quad 106.$$

In Eq. 106, $u = 0$, when $x = 0$; hence the constant of integration is zero.

Again, $u = 0$, when $x = l_1$, in Eq. 106. As that condition will make $x = z$, in the second term of the second member of Eq. 106, and $x = l_1$ in all the other terms, there will at once result:

$$C = \frac{l_1}{6}\left\{2H'f_1 - M_2 - Kz\left(1-\frac{z}{l_1}\right)\left(\frac{z^2}{l_1^2}+2\left(1-\frac{z}{l_1}\right)^2\right) - \frac{s'l_2^2}{4}\frac{a^3+r}{2a+3r}\right\} \quad . \quad . \quad . \quad . \quad . \quad 107.$$

The value of C given by Eq. 107 enables Eqs. 105 and 106 to be applied to any case.

If x lies between z and 0, the term $K\left(1-\frac{z}{l_1}\right)_o^z\frac{x^2}{2}$ in the second term of Eq. 105, and the corresponding term in the second member of Eq. 106, are to be taken as they stand; but x must be displaced by z in both those terms when x lies between l_1 and z. At the same time (*i.e.*, when the value of x is between z and 0), the term $Kz(x-z)\left(1-\frac{x+z}{2l_1}\right)_z^{l_1}$ in the second member of Eq. 105, as well as the corresponding term in the second member of Eq. 106, is to be omitted.

If, however, x lies between l_1 and z, the latter of the above terms in the second members of Eqs. 105 and 106 belonging between the same limits l_1 and z are to be taken as they stand.

When $x = z$ and $x = l_1$, those terms have respectively the following values:

For $x = z$:

$$K\left(1 - \frac{z}{l_1}\right)\frac{x^2}{2} = \frac{K z^2}{2}\left(1 - \frac{z}{l_1}\right)$$

For $x = l_1$:

$$K z (x - z)\left(1 - \frac{x+z}{2 l_1}\right) = K z (l_1 - z)\left(1 - \frac{l_1 + z}{2 l_1}\right) \qquad 108.$$

For $x = l_1$:

$$K z \frac{(x-z)^2}{2}\left(1 - \frac{2z+x}{3 l_1}\right) = \frac{K z}{3}(l_1 - z)^2\left(1 - \frac{z}{l_1}\right) \quad . \qquad 109.$$

If the spans l_1 and l_2 carry over their entire lengths the uniform moving loads p_1 and p_2 per linear foot of span, the moment M_0 (Eq. 97) has the value $M_0 = \frac{p_1}{2}(l_1 - x)\,x$, while M_2 and M_3 will be given by Eqs. 87 and 86, respectively. The two terms in which K is a factor in the second member of Eq. 104 will then be displaced by $\frac{p_1}{2}(l_1 - x)\,x$.

Furthermore, as $\frac{p_1}{2}\int_0^x (l_1 - x)\,x\,dx = \frac{p_1 x^2}{4}\left(l_1 - \frac{2x}{3}\right)$, this latter quantity will displace the two terms in which K is a factor in the second member of Eq. 105.

Again, as $\frac{p_1}{4}\int x_2\left(l_1 - \frac{2x}{3}\right)dx = \frac{p_1 x^3}{12}\left(l_1 - \frac{x}{2}\right)$, this latter quantity will displace the two terms in which K is a factor in the second term of Eq. 106.

Finally, since for $x = l_1$, $\frac{p_1 x^3}{12}\left(l_1 - \frac{x}{2}\right) = \frac{p_1 l_1^4}{24}$, the term in which K appears in the second member of Eq. 107 will be displaced by $\frac{p_1 l_1^2}{4}$.

With these simple changes Eqs. 104, 105, 106, and 107 may be at once used for uniform loads.

As $\frac{du}{dx}$ is the tangent of the inclination of the neutral surface of the truss to a horizontal, the point of greatest deflection will be found by placing $\frac{du}{dx} = 0$ and solving the resulting equation for x. The quantity so found must then be substituted in the general value of u, Eq. 106. This is a comparatively simple matter for uniform loading over the entire span, but not otherwise.

For a given manner and amount of loading, the deflections at a number of points may be found by Eq. 106, from which a deflection curve may be constructed, thus enabling the greatest deflection, the points of contraflexure or other desired points to be found by construction with sufficient accuracy.

All the preceding equations become directly applicable to span l_3 by simply changing l_1 to l_3, f_1 to f_3, and M_2 to M_3, and measuring x as well as z from the right-hand end of l_3.

Art. 50.—Points of Contraflexure in the Main or Central Continuous Span l_2.

Points of contraflexure in the main span l_2 of a continuous stiffening-truss are found precisely as in Art. 48 for the side spans l_1 and l_3.

Using the same notation as heretofore and considering a single load K at any panel distant z from the left end of the span, the values of M_0 in Eq. 92 of Art. 47, for the parts z and $l_2 - z$ of the span l_2, respectively, are as given by Eqs. 95 and 96 of Art. 48.

For the part z:

$$M_0 = K\frac{l_2 - z}{l_2}x = K\left(1 - \frac{z}{l_2}\right)x \quad . \quad . \quad . \quad 110.$$

For the part $l_2 - z$:

$$M_0 = K\frac{z}{l_2}(l_2 - x) = Kz - K\frac{z}{l_2}x \quad . \quad . \quad . \quad 111.$$

Also for a uniform load p_2 per linear foot covering the entire span:

$$M_0 = \frac{p_2}{2}(l_2 - x)\,x \quad . \quad . \quad . \quad . \quad . \quad 112.$$

If M_0, Eq. 110, be substituted in Eq. 92 of Art. 47, and if the value of the moment M, so found, be placed equal to zero, and if the resulting quadratic equation be solved for x, the position of the points of contraflexure will be given as follows:

$$x = \frac{l_2}{2}\left\{1 - \frac{M_3 - M_2 + K(l_2 - z)}{4H'f_2} \pm \sqrt{-\frac{1}{H'f_2}\left(M_2 + \frac{s'l_2^2}{4}\frac{a^3+r}{2a+3r}\right) + \left(1 - \frac{M_3 - M_2 + K(l_2-z)}{4H'f_2}\right)^2}\right\} \quad 113.$$

This equation shows that in general there will be two points of contraflexure in the part z of the span. Referring back to Eq. 88, and remembering that M_2 is negative in sign, it is seen that when the moment of the uniform suspender pull over a point of support is equal in amount (but opposite in sign) to M_2, the first term under the radical in Eq. 113 is equal to zero and there will be one point of contraflexure, the two values of x taking the form:

$$x = l_2\left(1 - \frac{M_3 - M_2 + K(l_2 - z)}{4H'f_2}\right) \text{ or } = 0 \quad . \quad 114.$$

There will also be but one value of x when the radical itself in Eq. 113 is equal to zero.

For the part $l_2 - z$ of the span, there must be substituted in Eq. 92 the value of M_0 given by Eq. 111. The solution of the quadratic equation resulting from that operation will yield:

$$x = \frac{l_2}{2}\left\{1 - \frac{M_3 - M_2 - Kz}{4H'f_2} \pm \sqrt{-\frac{1}{H'f_2}\left(Kz + M_2 + \frac{s'l_2^2}{4}\frac{a^3+r}{2a+3r}\right) + \left(1 - \frac{M_3 - M_2 - Kz}{4H'f_2}\right)^2}\right\} \quad . \quad . \quad . \quad . \quad 115.$$

This equation shows that in general there will be two points of contraflexure in the part $l_2 - z$ of the span. There evidently will be but one point of contraflexure in the special case for

which the radical in the second member of Eq. 115 becomes equal to zero.

If the uniform load p_2 per linear foot covers the entire span, M_0 given by Eq. 112 is to be substituted in Eq. 92; then placing its second member equal to zero, there will result:

$$x = \frac{l_2}{2}\left\{\left(\frac{8H'f_2 - 2M_3 + 2M_2 - p_2 l_2^2}{8H'f_2 - p_2 l_2^2}\right) \pm \sqrt{-\frac{M_2 + \frac{s' l_2^2}{4}\frac{a^2+r}{2a+3r}}{H'f_2 - \frac{p_2 l_2^2}{8}} + \left(\frac{8H'f_2 - 2M_3 + 2M_2 - p_2 l_2^2}{8H'f_2 - p_2 l_2^2}\right)^2}\right\} \quad . \quad . \quad 116.$$

There are, therefore, two points of contraflexure in this span.

These formulæ complete all which are needed to find points of contraflexure in any continuous span carrying any load under the prescribed conditions.

Art. 51.—Deflection in the Main or Central Continuous Span.

If the deflection at any point distant x from the left end be represented by u, as in Art. 49, the deflection equations for the main span l_2 may readily be written by the aid of Eq. 92 as follows:

$$EI_2\frac{d^2u}{dx^2} = M_2 + \frac{M_3 - M_2}{l_2}x + K\left(1 - \frac{z}{l_2}\right)_0^x x + Kz\left(1 - \frac{x}{l_2}\right)_z^{l_2} - 4H'f_2\left(1 - \frac{x}{l_2}\right)\frac{x}{l_2} + \frac{s' l_2^2}{4}\frac{a^2+r}{2a+3r} \quad . \quad . \quad 117.$$

Hence:

$$EI_2\frac{du}{dx} = M_2 x + \frac{M_3 - M_2}{2l_2}x^2 + K\left(1 - \frac{z}{l_2}\right)_0^x \frac{x^2}{2} + Kz\left(x - z - \frac{x^2 - z^2}{2l_2}\right)_z^{l_2} - H'\frac{2f_2}{l_2^2}\left(l_2 x^2 - \frac{2}{3}x^3\right) + \frac{s' l_2^2}{4}\frac{a^2+r}{2a+3r}x + C \quad . \quad 118.$$

Finally:

$$EI_2 u = M_2\frac{x^2}{2} + \frac{M_3 - M_2}{6l_2}x^3 + K\left(1 - \frac{z}{l_2}\right)_0^x \frac{x^3}{6} + Kz\frac{(x-z)^2}{2}\left(1 - \frac{2z+x}{3l_2}\right)_z^{l_2} - H'\frac{2f_2}{3l_2^2}\left(l_2 - \frac{x}{2}\right)x^3 + \frac{s' l_2^2}{8}\frac{a^2+r}{2a+3r}x^2 + Cx \quad . \quad 119.$$

Inasmuch as $u = 0$, when $x = 0$, the constant of integration for Eq. 119 is also zero. Again, in the same equation, $u = 0$ for $x = l_2$. Hence, since z must then take the place of x in the first term in which z appears in the second member of Eq. 119, and $x = l_2$ in the second such term in the same member of the same equation:

$$C = \frac{l_2}{6}\left\{2H'f_2 - 2M_2 - M_3 - Kz\left(1-\frac{z}{l_2}\right)\left(\frac{z^2}{l_2^2}+2\left(1-\frac{z}{l_2}\right)^2\right) - \frac{3s'l_2^2}{4}\frac{a^3+r}{2a+3r}\right\} \quad . \quad . \quad . \quad . \quad . \quad 120.$$

This value of C substituted in Eqs. 117, 118, and 119 will make them available for any application desired.

In making use of Eqs. 117, 118, and 119 reference must be made to the explanations given between Eqs. 107 and 109 of Art. 49, and in the paragraph following Eq. 109 of the values of the terms involving z and x for the different cases. With the full explanations found there, it is unnecessary to repeat them here.

Again, if the spans l_1, l_2, and l_3 carry the uniform loads p_1, p_2, and p_3 over their entire lengths, the non-continuous span moment $M_0 = \frac{p_2}{2}(l_2 - x)\,x$. The two moments, M_2 and M_3, will be given by Eqs. 87 and 86, respectively, as in Art. 49. Finally, the explanations beginning with the second paragraph, after Eq. 109, show completely the changes in the terms involving k in Eqs. 117, 118, and 119 to be made to adapt them to uniform loading over all the spans, it being only necessary in those explanations to displace p_1 and l_1 by p_2 and l_2.

The points of greatest deflection are to be found as before by placing $\frac{du}{dx} = 0$ and solving the resulting equation for x. Indeed, all the observations made in Art. 49 as to the points of greatest deflection, the construction of deflection curves, and points of contraflexures of the side spans l_1 and l_3 apply equally to the main span l_2.

The investigations of the preceding articles show fully the

character of the analytic treatment of continuous stiffening-trusses for determining points of contraflexure, deflections, and other features of elastic distortion caused by own weight or live loading.

Art. 52.—Bending Moments in a Non-Continuous Stiffening-Truss.

The value of the horizontal component H of the cable stress for non-continuous stiffening-trusses is given in Art. 46. With that value determined the bending moment in each of the three spans, l_1, l_2, and l_3, will be given by Eqs. 82, 83, and 84 of Art. 47.

As the truss is now non-continuous:

$$M_2 = M_3 = M' = M'' = 0 \quad . \quad . \quad . \quad . \quad 121.$$

Hence the bending moment in span l_2, for instance, is the same as that for either of the others except for the subscripts:

$$M = M_0 - H' \frac{4 f_2}{l_2^2} (l_2 - x) x \quad . \quad . \quad . \quad . \quad 122.$$

This equation obviously could be written directly without reference to Eqs. 82, 83, and 84. It is equally evident that it can be used for spans l_1 and l_3 by simply changing f_2 and l_2 to f_1, l_1, and f_3, l_3, respectively.

The general value of H' for non-continuous stiffening-trusses is given by Eq. 70 of Art. 46 by omitting all terms depending upon the dead load, *i.e.*, all terms in which k or w_0 appears. If those omissions be made, and if D be written for the denominator of the second member of Eq. 70:

$$H' = \frac{\frac{F_0}{I}\left\{\int_0^{l_2} M_b y \, dx + \frac{1}{r}\int_{0_1}^{l_1 l_1} M_b y_1 \, dx\right\}}{D} \quad . \quad 123.$$

For the special case of a stiffening-truss in the centre span

l_2 only, Eq. 74 of Art. 46 must be used. If D' is the denominator of the second member of Eq. 74, then for this case:

$$H' = \frac{\frac{F_0}{I}\int_0^{l_2} M_b\, y\, dx}{D'} \quad . \quad . \quad . \quad . \quad . \quad 124.$$

The general values of the integrals found in the second members of Eqs. 123 and 124 are given by Eqs. 71, 72, and 73 of Art. 46, or in the special case of a uniform load by Eqs. 59 and 60 of Art. 44.

That part of the horizontal stress H represented by H'', and due to the dead load, is found for the two cases of non-continuous trusses by making the numerators of Eqs. 123 and 124 equal to zero in the second members of Eqs. 70 and 74 of Art. 46, respectively, without changing the denominators D and D'

Uniform Load.

If it be supposed that the stiffening-truss is traversed by the uniform moving load p_2 per linear foot, that load extending the distance x_1, from one end of the span, Eq. 124 will take the form, by the aid of Eq. 59:

$$H' = \frac{\frac{F_0}{I}\frac{f_2\, p_2\, l_2^2}{6}\left(\frac{x_1^3}{l_2^3} - \frac{x_1^4}{l_2^4} + \frac{2}{5}\frac{x_1^5}{l_2^5}\right)}{D'} \quad . \quad . \quad 125.$$

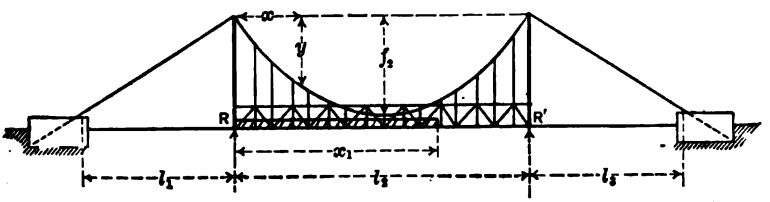

FIG. 15.

The reaction R at the left end of the span, Fig. 15, assuming an ordinary simply supported beam of length l_2, is $R = p_2\left(x_1 - \frac{x_1^2}{2l_2}\right)$, while the reaction R' at the right end of that span is

$R' = \frac{p_2 x_1^2}{2 l_2}$. Hence the moving load bending moment M_0 for the loaded part x_1 of the span will be:

$$M_0 = R x - \frac{p_2 x^2}{2} = p_2 \left\{ \left(x_1 - \frac{x_1^2}{2 l_2} \right) x - \frac{x^2}{2} \right\} \quad . \quad 126.$$

By substitution in Eq. 122:

$$M = p_2 \left\{ \left(x_1 - \frac{x_1^2}{2 l_2} \right) x - \frac{x^2}{2} \right\} - H' \frac{4 f_2}{l_2^2} (l_2 x - x^2) \quad . \quad 127.$$

To find the point of greatest moment in the loaded part x_1 of the truss:

$$\frac{dM}{dx} = p_2 \left(x_1 - \frac{x_1^2}{2 l_2} \right) - p_2 x - H' \frac{4 f_2}{l_2} + H' \frac{8 f_2}{l_2^2} = 0 \quad . \quad 128.$$

By introducing the value of H^1 from Eq. 125 and solving this equation:

$$x = l_2 \frac{\frac{F_0}{ID^1} \frac{2 f_2^2 l_2}{3} \left(\frac{x_1^2}{l_2^2} - \frac{x_1^4}{l_2^4} + \frac{2}{5} \frac{x_1^5}{l_2^5} \right) - \left(\frac{x_1}{l_2} - \frac{1}{2} \frac{x_1^2}{l_2^2} \right)}{\frac{F_0}{ID^1} \frac{4 f_2^2 l_2}{3} \left(\frac{x_1^2}{l_2^2} - \frac{x_1^4}{l_2^4} + \frac{2}{5} \frac{x_1^5}{l_2^5} \right) - 1.} \quad . \quad . \quad 129.$$

This value of x substituted in Eq. 127 will give the greatest bending moment in the loaded part x_1 of the span, that moment being such as to produce compression in the upper chord and tension in the lower chord. Making that substitution, there will result the following general expression for the greatest bending moment produced by the uniform moving load, $p_2 x_1$, extending from one end of the span:

$$M_m = -\tfrac{1}{2} p_2 l_2^2 \frac{\left\{ \frac{F_0}{ID'} \frac{2 f_2^2}{3} \left(\frac{x_1^2}{l_2^2} - \frac{x_1^4}{l_2^4} + \frac{2}{5} \frac{x_1^5}{l_2^5} \right) - \left(\frac{x_1}{l_2} - \frac{1}{2} \frac{x_1^2}{l_2^2} \right) \right\}^2}{\frac{F_0}{ID'} \frac{4 f_2^2 l_2}{3} \left(\frac{x_1^2}{l_2^2} - \frac{x_1^4}{l_2^4} + \frac{2}{5} \frac{x_1^5}{l_2^5} \right) - 1.} \quad . \quad 130.$$

The maximum bending moment in the entire span will be found by taking the first derivative of M_m, a function of x_1, from Eq. 130 and placing it equal to zero. The resulting value of x_1 will give the length of loading for that maximum moment, and if

the same value of x_1 be placed in Eq. 130, the maximum moment itself will at once be found. Eq. 129 will then locate the point of maximum bending. This operation, however, is too complicated to be easily completed and other procedures are preferable.

If the ratio $\frac{x_1}{l_2}$ be varied from 0 to 1 so as to cover the entire span and if the corresponding values of x (Eq. 129) and M_m (Eq. 130) be found, a greatest moment curve may be constructed with the abscissa x and ordinate M_m, which will show completely the greatest possible bending moment at every point of the span, including the maximum of all. This should be done whenever the law of loading may be that under consideration. The computations for both x and M_m, as well as for H', will be much aided by the following table or by the curves which may be constructed from it.

$\frac{x_1}{l_2}$	$\frac{x_1^3}{l_2^3} - \frac{x_1^4}{l_2^4} + \frac{2}{5}\frac{x_1^5}{l_2^5}$	$\frac{x_1}{l_2} - \frac{1}{2}\frac{x_1^2}{l_2^2}$
.05	.002493275	.04875
.1	.009904	.095
.15	.022024125	.13875
.2	.038528	.18
.25	.058984375	.21875
.3	.082872	.255
.35	.10959462	.28875
.4	.138496	.32
.45	.168874875	.34875
.5	.2	.375
.55	.231125125	.39875
.6	.261504	.42
.65	.290405375	.43875
.7	.317128	.455
.75	.341015625	.46875
.8	.361472	.48
.85	.377975875	.48875
.9	.390696	.495
.95	.397502405	.49875
1.	.4	.5

The negative moving load moments, existing in the unloaded part of the span ($l_2 - x_1$), with the uniform moving load taken as

already assumed, are readily written as follows, remembering that x must not be taken less than x_1:

$$M_0 = R'(l_2 - x) = \frac{p_2 x_1^2}{2 l_2}(l_2 - x) \quad . \quad . \quad . \quad 131.$$

By substitution in Eq. 122:

$$M = \frac{p_2 x_1^2}{2 l_2}(l_2 - x) - H' \frac{4 f_2}{l_2^2}(l_2 x - x^2) \quad . \quad . \quad 132.$$

Proceeding as before to find the point of greatest negative moment, *i.e.*, in the unloaded part of the beam:

$$\frac{dM}{dx} = -\frac{p_2 x_1^2}{2 l_2} - H' \frac{4 f_2}{l_2} + H' \frac{8 f_2}{l_2^2} x = 0 \quad . \quad . \quad 133.$$

Hence:

$$x = \frac{l_2}{2}\left(1 + \frac{D'}{\frac{F_0}{I}\frac{4 f_2^2 l_2}{3}\left(1 - \frac{x_1^2}{l_2^2} + \frac{2}{5}\frac{x_1^3}{l_2^3}\right)}\right) \quad . \quad 134.$$

By placing this value of x in Eq. 132, and using Eq. 125 for H', the greatest negative moment in any unloaded part $(l_2 - x_1)$ of a span will be found, x never being less than x_1. This operation will give:

$$M'_m = \frac{p_2 x_1^2}{4}\left(1 - \frac{D'}{\frac{F_0}{I}\frac{8 f_2^2 l_2}{3}\left(1 - \frac{x_1^2}{l_2^2} + \frac{2}{5}\frac{x_1^3}{l_2^3}\right)} - \frac{\frac{F_0}{I}\frac{2 f_2^2 l_2}{3}\left(1 - \frac{x_1^2}{l_2^2} + \frac{2}{5}\frac{x_1^3}{l_2^3}\right)}{D'}\right) \quad . \quad . \quad . \quad . \quad 135.$$

This equation, like Eq. 130, may be differentiated in respect to x_1 as the variable in the search for the greatest value of M'_m in the entire span. By placing the first derivative equal to zero and solving that equation for x_1, then placing that value in Eq. 135, the greatest moment M'_m in the entire span will at once result, but that operation is too complicated to be practicable, and, as in the preceding case, another procedure is preferable.

Again, varying $\frac{x_1}{l_2}$ from zero to unity so as to cover the entire

span, the moment M'_m and its location x will be found by Eqs. 135 and 134, respectively. A greatest-moment curve may then be constructed with the abscissa x and moment M'_m, which will show the greatest moments in the unloaded parts of the span, including the greatest in the entire span. This operation will be expedited by the use of the values given in the following table:

$\frac{x_1}{l_1}$	$1-\frac{x_1^2}{l_2^2}+\frac{2}{5}\frac{x_1^3}{l_2^3}$	$\frac{x_1}{l_2}$	$1-\frac{x_1^2}{l_2^2}+\frac{2}{5}\frac{x_1^3}{l_2^3}$
.05	.99755	.55	.76405 (?)
.1	.9904	.6	.7264
.15	.97885	.65	.68735
.2	.9632	.7	.6472
.25	.94375	.75	.60625
.3	.9208	.8	.5648
.35	.89465	.85	.52315
.4	.8656	.9	.4816
.45	.83395	.95	.44045
.5	.8	1.	.4

These numerical values will enable both x (Eq. 134), and M'_m (Eq. 135), to be readily computed or constructed after F_0 and I are either tentatively or finally determined. The quantity D' is to be found by the aid of Eq. 74, Art. 46, as it is the denominator of the second member of that equation.

Art. 53.—Deflection of the Non-Continuous Stiffening-Truss.

The deflection of the non-continuous stiffening-truss, due to flexure only, may be at once found by the aid of the preceding formulæ.

Single Load K at Any Point.

The general equation for the bending moment for this case is Eq. 122 of Art. 52. In the most general case, the simple beam moment M_0 for a single load K at the distance z from the left end of the span is, for the segment of the span o to z:

$$M_0 = K\frac{l_2 - z}{l_2}x = K\left(1 - \frac{z}{l_2}\right)x \quad . \quad . \quad . \quad 136.$$

and for the segment z to l:

$$M_0 = K\frac{z}{l_2}(l_2 - x) = K\left(1 - \frac{x}{l_2}\right)z \quad . \quad . \quad 137.$$

From o to z, therefore, the general bending moment will take the form, Eq. 122:

$$M = K\left(1 - \frac{z}{l_2}\right)x - H'\,4f_2\left(1 - \frac{x}{l_2}\right)\frac{x}{l_2}^{*} \quad . \quad 138.$$

And from z to l_2:

$$M = K\left(1 - \frac{x}{l_2}\right)z - H'\,4f_2\left(1 - \frac{x}{l_2}\right)\frac{x}{l_2} \quad . \quad . \quad 139.$$

The values of H' for the two cases of stiffening-trusses in all three spans or in the main or central span only are given in Eqs. 123 and 124 of Art. 52.

If u is the deflection, then for o to z:

$$M = EI\frac{d^2u}{dx^2} = K\left(1 - \frac{z}{l_2}\right)x - H'\,4f_2\left(1 - \frac{x}{l_2}\right)\frac{x}{l_2} \quad . \quad 140.$$

And from z to l_2:

$$EI\frac{d^2u}{dx^2} = K\left(1 - \frac{x}{l_2}\right)z - H'\,4f_2\left(1 - \frac{x}{l_2}\right)\frac{x}{l_2} \quad . \quad 141.$$

If the first integration be made in Eqs. 140 and 141, C being the constant of integration in the first of those equations and C' the constant in the second, and if the two results be placed equal to each other after making $x = z_1$ in each, there will result:

$$C' = C - K\frac{z^2}{2}$$

Hence, for the part o to z of the span:

$$EI\frac{du}{dx} = K\frac{l_2^2}{2}\left(1 - \frac{z}{l_2}\right)\frac{x^2}{l_2^2} - H'\,4f_2 l_2\left(\frac{1}{2}\frac{x^2}{l_2^2} - \frac{1}{3}\frac{x^3}{l_2^3}\right) + C \quad . \quad 142.$$

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*Tabular values of $\left(1 - \frac{x}{l}\right)\frac{x}{l}$ may be found in Art. 11.

and for the part s to l_2:

$$EI\frac{du}{dx}=Kl_2^2\frac{z}{l_2}\left(\frac{x}{l_2}-\frac{1}{2}\frac{x^2}{l_2^2}\right)^*-H'4f_2l_2\left(\frac{1}{2}\frac{x^2}{l_2^2}-\frac{1}{3}\frac{x^3}{l_2^3}\right)+C' \quad . \quad 143.$$

Integrating Eq. 142 and remembering that $u = 0$ for $x = 0$:

$$EIu=K\frac{l_2^3}{6}\left(1-\frac{z}{l_2}\right)\frac{x^3}{l_2^3}-H'\frac{4f_2l_2^2}{6}\left(\frac{x^3}{l_2^3}-\frac{1}{2}\frac{x^4}{l_2^4}\right)+Cx \quad . \quad 144.$$

Also $u = 0$ for $x = l_2$; hence integrating Eq. 143 between the limits x and l_2:

$$EIu=Kl_2^3\frac{z}{l_2}\left(\frac{1}{2}\frac{x^2}{l_2^2}-\frac{1}{6}\frac{x^3}{l_2^3}-\frac{1}{3}\right)-H'\frac{4f_2l_2^2}{6}\left(\frac{x^3}{l_2^3}-\frac{1}{2}\frac{x^4}{l_2^4}-\frac{1}{2}\right)$$
$$+C'l_2\left(\frac{x}{l_2}-1\right) \quad . \quad . \quad . \quad . \quad . \quad 145.$$

Making $x = z$ in Eqs. 144 and 145 and then placing the second members of those equations equal to each other:

$$\left.\begin{aligned}C&=-K\frac{l_2^2}{6}\frac{z}{l_2}\left(\frac{z^2}{l_2^2}-\frac{3z}{l_2}+2\right)+H'\frac{f_2l_2}{3}\\ C'&=C-K\frac{z^2}{2}=-K\frac{l_2^2}{6}\frac{z}{l_2}\left(\frac{z^2}{l_2^2}+2\right)+H'\frac{f_2l_2}{3}\end{aligned}\right\} \quad . \quad 146.$$

The values of C and C' make equations 142, 143, 144, and 145 immediately available for application.

If there are a number of isolated loads K resting on the truss at different points in the span, the resultant inclination of the neutral surface (Eqs. 142 and 143) and the resultant deflection (Eqs. 144 and 145) of any point x will be found by simply adding the separate results for each single load K for the same point.

The following tabulations will greatly aid in the application of the preceding formulæ:

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* Tabular values of $\left(\frac{x}{l}-\frac{1}{2}\frac{x^2}{l^2}\right)$ may be found on page 150 of this article.

$\frac{x}{l}$	$\frac{1}{2}\frac{x^2}{l^2}-\frac{1}{3}\frac{x^3}{l^3}$	$\frac{1}{2}\frac{x^2}{l^2}-\frac{1}{6}\frac{x^3}{l^3}-\frac{1}{3}$	$\frac{x^3}{l^3}-\frac{1}{2}\frac{x^4}{l^4}$
.05	.001,208,333	— .332,104,167	.000,121,875
.1	.004,666,667	— .328,5	.000,95
.15	.010,125	— .322,645,833	.003,121,875
.2	.017,333,333	— .314,666,667	.007,2
.25	.026,041,667	— .304,687,5	.013,671,875
.3	.036	— .292,833,333	.022,95
.35	.046,958,333	— .279,229,167	.035,371,875
.4	.058,666,667	— .264,	.051,2
.45	.070,875	— .247,270,333	.070,621,875
.5	.083,333,333	— .229,166,667	.093,75
.55	.091,791,666	— .213,812,5	.120,621,875
6	.108	— .189,333,333	.151,2
.65	.119,708,333	— .167,854,167	.185,371,875
.7	.130,666,666	— .145,5	.222,95
.75	.140,625	— .122,395,833	.263,671,875
.8	.149,333,333	— .098,666,667	.307,2
.85	.156,541,666	— .074,437,500	.353,121,875
.9	.162	— .049,833,333	.400,95
.95	.165,458,333	— .024,979,167	.450,121,875
1.00	.166,666,666	— .00	.5

$\frac{x}{l}$	$\frac{x^3}{l^3}-\frac{1}{2}\frac{x^4}{l^4}-\frac{1}{2}$	$\frac{x}{l}\left(\frac{x^2}{l^2}-\frac{3x}{l}+2\right)$	$\frac{x}{l}\left(\frac{x^2}{l^2}+2\right)$
.05	— .499,878,125	.092,6	.100,125
.1	— .499,05	.171	.201
.15	— .496,878,125	.235,875	.303,375
.2	— .492,8	.288	.408
.25	— .486,328,125	.328,125	.515,625
.3	— .477,05	.357	.627
.35	— .464,628,125	.375,375	.742,875
.4	— .448,8	.384	.864
.45	— .429,378,125	.383,625	.991,125
.5	— .406,25	.375	1.125
.55	— .379,378,125	.358,875	1.266,375
.6	— .348,8	.336	1.416
.65	— .314,628,125	.307,125	1.574,625
.7	— .277,05	.273	1.743
.75	— .236,328,125	.234,375	1.921,875
.8	— .192,8	.192	2.112
.85	— .146,878,125	.146,625	2.314,125
.9	— .099,05	.099	2.529
.95	— .049,878,125	.049,875	2.757,375
1.00	— .00	0.00	3.00

It is evident that the formulæ of this article apply at once to either of the side spans, l_1 and l_3, by simply changing l_2 to either l_1 or l_3.

Uniform Loading from One End of Span.

If the uniform loading p_2 per linear foot extend from one end of the span l_2 to the distance x_1, as shown in Fig. 15 of Art. 52, the bending moment M, for the loaded part of the span, has the value given by Eq. 127 of the same article. Hence:

$$E\,I\frac{d^2u}{dx^2} = p_2\left\{\left(x_1 - \frac{x_1^2}{2l_2}\right)x - \frac{x^2}{2}\right\} - H'\frac{4f_2}{l_2^2}(l_2\,x - x^2) \quad . \quad 147.$$

By using Eq. 132 of Art. 52 for the unloaded part of the span, x_1 to l_2:

$$E\,I\frac{d^2u}{dx^2} = \frac{p_2 x_1^2}{2l_2}(l_2 - x) - H'\frac{4f_2}{l_2^2}(l_2\,x - x^2) \quad . \quad . \quad 148.$$

Integrating Eq. 147:

$$E\,I\frac{du}{dx} = p_2\left\{\left(x_1 - \frac{x_1^2}{2l_2}\right)\frac{x^2}{2} - \frac{x^3}{6}\right\} - H'\frac{4f_2}{l_2^2}\left(l_2\frac{x^2}{2} - \frac{x^3}{3}\right) + C \quad . \quad 149.$$

In this equation C is the constant of integration. A second integration gives, remembering that for $x = 0$, $u = 0$:

$$E\,I\,u = p_2\left\{\left(x_1 - \frac{x_1^2}{2l}\right)\frac{x^3}{6} - \frac{x^4}{24}\right\} - H'\frac{4f_2}{l_2^2}\left(l_2\frac{x^3}{6} - \frac{x^4}{12}\right) + Cx \quad . \quad 150.$$

By the first integration of Eq. 148:

$$E\,I\frac{du}{dx} = \frac{p_2 x_1^2}{2l_2}\left(l_2\,x - \frac{x^2}{2}\right) - H'\frac{4f_2}{l_2^2}\left(l_2\frac{x^2}{2} - \frac{x^3}{3}\right) + C' \quad . \quad 151.$$

Again, integrating between the limits x and l_2, remembering that $u = 0$ for $x = l_2$:

$$E\,I\,u = \frac{p_2 x_1^2}{2l_2}\left(l_2\frac{x^2}{2} - \frac{x^3}{6} - \frac{l_2^3}{3}\right) - H'\frac{2f_2}{3l_2^2}\left(l_2\,x^3 - \frac{x^4}{2} - \frac{l_2^4}{2}\right) + C'\,x - C'\,l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad 152.$$

Making $x = x_1$, in Eqs. 150 and 152, and placing the second members of those equations equal to each other:

$$C'\left(1 - \frac{x_1}{l_2}\right) = \frac{p_2 l_2^3}{8}\frac{x_1^4}{l_2^4} - \frac{p_2 l_2^3}{6}\frac{x_1^2}{l_2^2} + \frac{H' l_2^2 f_2}{3\,l_2} - C\frac{x_1}{l_2} \quad . \quad 153.$$

Making $x = x_1$ in Eqs. 149 and 151, and placing their second members equal to each other:

$$C' = -\frac{p_2 x_1^3}{6} + C \quad . \quad . \quad . \quad . \quad . \quad 154.$$

Eliminating C between Eqs. 153 and 154:

$$C' = -\frac{p_2 l_2^3}{6}\left(\frac{x_1^2}{l_2^2} + \frac{x_1^4}{l_2^4}\right) + H'\frac{l_2^2 f_2}{3 l_2} \quad . \quad . \quad . \quad 155.$$

If the loading covers the entire span, $x_1 = l_2$ and:

$$C' = -\frac{5\, p_2 l_2^3}{24} + \frac{H' l_2^2 f_2}{3 l_2} \quad . \quad . \quad . \quad . \quad 155a.$$

Eq. 154 will then give:

$$C = \frac{p_2 l_2^3}{6}\left(-\frac{x_1^2}{l_2^2} + \frac{x_1^3}{l_2^3} - \frac{x_1^4}{4 l_2^4}\right) + H' l_2^2 \frac{f_2}{3 l_2} \quad . \quad 156.$$

The following table will simplify and shorten the application of the second members of Eqs. 155 and 156:

$\frac{x_1}{l}$	$\frac{x_1^2}{l^2} + \frac{x_1^4}{l^4}$	$-\frac{x_1^2}{l^2} + \frac{x_1^3}{l^3} - \frac{x_1^4}{4l^4}$
.05	.002,506,25	— .002,376,562,5
.1	.010,1	— .009,025
.15	.023,006,25	— .019,251,562,5
.2	.041,6	— .032,4
.25	.066,406,25	— .047,851,562,5
.3	.098,1	— .065,025
.35	.137,506,25	— .083,391,562,5
.4	.185,6	— .102,400
.45	.243,506,25	— .121,626,562,5
.5	.312,5	— .140,625
.55	.404,006,25	— .156,501,562,5
.6	.489,6	— .176,4
.65	.601,006,25	— .192,501,562.5
.7	.730,1	— .207,025
.75	.878,906,25	— .219,726,562,5
.8	1.049,6	— .230,4
.85	1.244,506,25	— .238,876,562,5
.9	1.466,1	— .245,025
.95	1.717,006,25	— .248,751,562,5
1.00	2.0	— .25

Eqs. 155 and 156 make known the values of the constants C' and C of integration for any uniform loading $p_2 x_1$. Those two quantities being known, Eqs. 149, 150, 151, and 152 will at once give the inclination $\frac{du}{dx}$ of the neutral surface of the truss and the deflection u of the latter at any point of either the loaded or unloaded segment of the span, after dividing the second member of each equation by EI.

If the load covers the entire span, $x_1 = l_2$ in Eq. 149, and:

$$EI\frac{du}{dx} = p_2\left(\frac{l_2 x^2}{4} - \frac{x^3}{6}\right) - H'\frac{4f_2}{l_2^2}\left(l_2\frac{x^2}{2} - \frac{x^3}{3}\right) + C \quad . \quad 157.$$

For the same condition Eq. 150 gives:

$$EIu = p_2\left(\frac{l_2 x^3}{12} - \frac{x^4}{24}\right) - H'\frac{4f_2}{l_2^2}\left(l_2\frac{x^3}{6} - \frac{x^4}{12} + Cx\right) . \quad 158.$$

Also, for the same condition:

$$C = -\frac{p_2 l_2^3}{24} + \frac{H' l_2^2 f_2}{3 l_2} \quad . \quad . \quad . \quad . \quad 159.$$

At the centre of the span $x = \frac{l_2}{2}$ and $EI\frac{du}{dx} = 0$. Also if s' be the suspender pull per linear unit (foot) of span, $H' = \frac{s' l_2^2}{8 f_2}$. Hence:

$$EI u_c = \frac{5 l_2^3}{384}\left(-p_2 l_2 + 8 H'\frac{f_2}{l_2}\right) = -\left(p_2 - s'\right)\frac{5 l_2^4}{384} \quad . \quad 160.$$

The equations of this article, thus completed, are sufficient to compute the deflections and slope of the neutral surface of the truss at all points for any loading whatever.

Art. 54.—Shears in Continuous Stiffening-Trusses.

Inasmuch as the total transverse shear in any beam can be at once found when the general expression for the bending moment is known, the shears in the continuous stiffening-trusses considered in this chapter can be immediately written from Eqs.

91, 92, and 93 of Art. 47, the notation remaining as heretofore. The expressions will first be given for a single load K and subsequently for a uniform load.

The bending moment M_0 for a single load K at the distance z from the left-hand end of a non-continuous span is given for both segments of the span by Eqs. 95 and 96 of Art. 47.

In writing these expressions for the shears in the three spans, l_1, l_2, and l_3, it is to be observed that there are no towers at the anchorages at the extremities of the spans l_1 and l_3, the cables being horizontal at the latter points. If therefore $l_1 = l_3$, and if h_1 is the height of the highest point of the cable at the tower at either extremity of the central span above the cable adjacent to the anchorage, so that $\tan \beta = \frac{h_1}{l_1}$, the first of Eqs. 23, Art. 42, shows that $H' \tan \beta$ must appear as a term in Eqs. 161 and 163, but not in Eq. 162, as the towers of the span l_2 are of equal height.

If S_1, S_2, and S_3 are the shears in the spans, l_1, l_2, and l_3, then, since in general $S = \frac{dM}{dx}$ for horizontal beams, with vertical loading only, there will result after adding $H' \tan \beta$ to $\frac{dM}{dx}$:

For span l_1:

$$S_1 = \frac{M_2}{l_1} + \left[K_1\left(1 - \frac{z}{l_1}\right)\right]_0^s \text{or} \left[-K_1 \frac{z}{l_1}\right]_{l_1} - H' \frac{4 f_1}{l_1^2}(l_1 - 2x) + H' \tan \beta + \frac{s' l_2^2}{4 l_1} \frac{a^3 + r}{2a + 3r} \quad . \quad . \quad . \quad 161.$$

For span l_2:

$$S_2 = \frac{M_3 - M_2}{l_2} + \left[K_2\left(1 - \frac{z}{l_2}\right)\right]_0^s \text{or} \left[-K_2 \frac{z}{l_2}\right]_{l_2}^s - H' \frac{4 f_2}{l_2^2}(l_2 - 2x) \quad . \quad . \quad . \quad . \quad . \quad 162.$$

For span l_3:

$$S_3 = -\frac{M_3}{l_3} + \left[K_3\left(1 - \frac{z}{l_3}\right)\right]^s \text{or} \left[-K_3 \frac{z}{l_3}\right] - H' \frac{4 f_3}{l_3^2}(l_3 - 2x) + H' \tan \beta - \frac{s' l_2^2}{4 l_3} \frac{a^3 + r}{2a + 3r} \quad . \quad . \quad . \quad . \quad 163.$$

It may be observed that these equations are identical with Eqs. 24, 25, and 26 of Art. 42, H' being the horizontal component of cable stress due to the moving load.

The values of M_2 and M_3 are to be taken from Eqs. 50 and 51 of Art. 44, where they have their most general forms, which can readily be adapted to any special loading on one or more spans.

Uniform Loading from One End of Span.

If the uniform loading, p_1 per linear unit (*i.e.*, foot), extend from the left end of each span for the distance x_1, the moment M_0 will have the value, between o and x, for the span l_1:

$$M_0 = p_1 \left\{ \left(x_1 - \frac{x_1^2}{2l_1} \right) x - \frac{x^2}{2} \right\} \quad . \quad . \quad . \quad 164.$$

And for the unloaded part of the span l_1, *i.e.*, for values of x lying between x_1 and l_1:

$$M_0 = \frac{p_1 x_1^2}{2 l_1} (l_1 - x) \quad . \quad . \quad . \quad . \quad . \quad 165.$$

For the spans l_2 and l_3, the subscript 1 in Eqs. 164 and 165 should be changed to 2 and 3 respectively, but no other change is needed.

If the uniform load p_1 should pass on the span from the right-hand end, it is only necessary to consider x and x_1 measured from that right-hand end. One or more spans may be subject to such a condition of loading, while the others carry moving load extending from the left end of the span or spans.

Using Eq. 164:

$$\frac{d M_0}{d x} = p_1 \left\{ \left(x_1 - \frac{x_1^2}{2l_1} \right) - x \right\} \quad . \quad . \quad . \quad . \quad 166.$$

Using Eq. 165:

$$\frac{d M_0}{d x} = - \frac{p_1 x_1^2}{2 l_1} \quad . \quad . \quad . \quad . \quad . \quad 167.$$

For the uniform load, therefore, the second member of Eq.

166, with the proper subscript, is to be substituted in place of the typical term $\left[K\left(1-\frac{z}{l}\right)\right]_0^z$ in Eqs. 161, 162, and 163. Similarly, the second member of Eq. 167 is to be substituted in place of the typical term $\left[-K\frac{z}{l}\right]^z$ in Eqs. 161, 162, and 163. Also, the values of M_2 and M_3 are to be taken from Eqs. 86 and 87 of Art. 47.

The changes thus indicated enable Eqs. 161, 162, and 163 to be used directly for a continuous moving load passing on any span or spans in either direction.

If, for example, a uniform moving load should cover all of l_1 and half of l_2, $x_1 = l_1$ in Eq. 166, and $\frac{dM_0}{dx} = p_1\left(\frac{l_1}{2} - x\right)$. At the same time, for the span l_2, $\frac{dM_0}{dx} = p_2\left(\frac{3\,l_2}{8} - x\right)$ for the loaded half, and for the unloaded part:

$$\frac{d\,M_0}{d\,x} = -\frac{p_2\,l_2}{8}.$$

In this case, the terms in which K_3 appears in Eq. 163 would be zero if the span l_3 carried no moving load.

Art. 55.—Shears in Non-Continuous Stiffening-Trusses.

The general value of the bending moment in the non-continuous stiffening-truss is shown by Eq. 122 of Art. 52. By giving M_0 its proper values for the two parts of a span in which a single load K is placed at the distance z from one end, that equation becomes, after omitting the subscripts of K for non-continuous trusses:

$$M = \left[K_2\left(1-\frac{z}{l_2}\right)x\right]^z \text{ or } \left[K_2\,z\left(1-\frac{x}{l_2}\right)\right]_z^{l_2} - H'\frac{4\,f_2}{l_2^2}(l_2-x)x \;.\; 168.$$

In this equation H' is the horizontal component of the cable stress due to moving load and it is given by Eqs. 123 and 124 of Art. 52.

The shear will then become:

$$S_2 = \frac{dM}{dx} = \left[K\left(1 - \frac{z}{l_2}\right)\right]_z \text{or} \left[-\frac{Kz}{l_2}\right]_z^{l_2} - H'\frac{4f_2}{l_2}\left(1 - \frac{2x}{l_2}\right) . \quad 169.$$

For the reasons given in establishing Eqs. 161 and 163, Eq. 169 must be modified if the towers at the extremities of a span are of unequal heights, as in the case of the side spans l_1 and l_2.

If the height of the cable at one end of the main or central span above the cable at the other extremity of the side span (usually adjacent to the anchorage) is h_1, and if $\frac{h_1}{l_1} = tan\ \beta_1$, then will the shear for the side span l_1, in which is supposed to be a non-continuous stiffening-truss carried by suspenders from the cable of the side span, be given by the equation:

$$S_1 = \left[K\left(1 - \frac{z}{l_1}\right)\right]_z \text{or} \left[-\frac{Kz}{l_1}\right]_z^{l_1} - H'\frac{4f_1}{l_1^2}(l_1 - 2x) + H'tan\beta_1 . \quad 170.$$

Eq. 170 is made applicable to span l_3 by simply changing the subscript 1 to 3.

If there are a number of single concentrations, K on the span, then, in general, the sign of summation Σ must precede each term involving K in Eqs. 169 and 170. In each case H' is the horizontal component of cable stress due either to a single load K or to such number of single loads as may rest concurrently on the truss.

Uniform Loading from One End of the Span.

If the truss be traversed by the uniform loading p per lineal foot for the distance x_1 from one end, each of the terms containing K in Eqs. 169 and 170 must be displaced by the second member of Eq. 166 for the loaded part of the span, or by the second member of Eq. 167 for the unloaded part. The explanations accompanying and following Eqs. 164, 165, 166, and 167 make clear the only changes needed to adapt Eqs. 169 and 170 to any uniform moving load from either end of the span.

Art. 56.—Non-Continuous Stiffening-Trusses—Effect of Loading One Span on Others.

If non-continuous stiffening-trusses are used in the side spans, moving load on any one of the three spans will produce suspender and cable stresses in the others with attendant bending moments and shears.

Let it be supposed in Fig. 16 that the centre span l_2 carries any moving load whatever causing a cable stress in that span with horizontal component H', to be found by Eq. 123 or Eq.

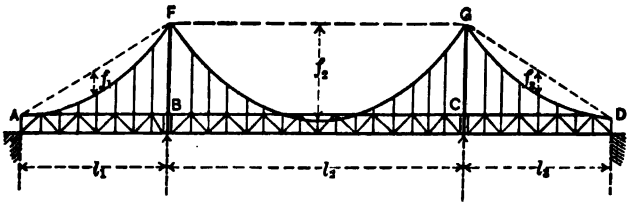

FIG. 16.

124 of Art. 52. This condition will induce moving-load cable stresses in spans l_1 and l_3 with the same horizontal component H' and, hence, with the same suspender pull s' per linear unit of span.

If the centre ordinates are f_1, f_2, and f_3, as shown in Fig. 16, since the three curves have the same parameters:

$$H' = \frac{s' l_1^2}{8 f_1} = \frac{s' l_2^2}{8 f_2} = \frac{s' l_3^2}{8 f_3} \quad . \quad . \quad . \quad . \quad 171.$$

Bending Moments.

The bending moments, deflections, and shears in the centre span (*i.e.*, in the span carrying the moving load) will be found by the equations of Arts. 52, 53, and 54, but the moving load effects in the spans l_1 and l_3 are simply the uniform upward pulls s'. Hence the moving-load bending moments will be:

$$M'_1 = -\frac{s'x}{2}(l_1 - x) \text{ and } M'_3 = -\frac{s'x}{2}(l_3 - x) \quad . \quad . \quad 172.$$

In these equations:

$$s' = \frac{8 f_1 H'}{l_1^2} = \frac{8 f_2 H'}{l_2^2} = \frac{8 f_3 H'}{l_3^2} \quad . \quad . \quad . \quad 173.$$

The negative sign is used in the second members of Eqs. 172 because the bending of the two stiffening-trusses l_1 and l_3 is such as to make them convex upward, *i.e.*, opposite to the bending caused by the ordinary vertical load acting downward.

Shears.

Similarly the shears in the spans l_1 and l_3 are, respectively:

$$S'_1 = -s'\left(\frac{l_1}{2} - x\right) \text{ and } S'_3 = -s'\left(\frac{l_3}{2} - x\right) \quad . \quad 174.$$

The negative sign again indicates a direction of shear opposite to that caused by the ordinary vertical downward load.

At the ends of a span, as l_1, where $x = 0$:

$$S'_1 = -\frac{s' l_1}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 175.$$

Eq. 175 gives the end shear, *i.e.*, the amount of anchorage which must be provided at each end of the span.

Deflections.

The deflections in the spans l_1 and l_2 are to be written by the usual formulæ for non-continuous trusses or beams loaded uniformly over the entire span, and it is unnecessary to repeat them here. It is only to be remembered that the deflection in this case is upward instead of downward.

Moving Load in a Side Span.

If the moving load is in a side span, the formulæ of Arts. 52, 53, and 54 are to be used for that span, while those established above in this article are to be used for the centre span and the other side span. It is only required to change the subscripts to adapt the latter to this case, s' being found as before by Eq. 173.

Art. 57.—Influence Lines for Moments and Shears—Continuous Trusses.

Moments.

The equations of Art. 47 are immediately applicable to the construction of influence lines. The general values of the moments M_2 and M_3 over the supports of an ordinary continuous beam carrying moving load are given by Eqs. 14 and 15 of Art. 42. If, as is usual, the influence line for a single load P in any one span is being constructed, all terms in the numerators of Eqs. 14 and 15 of Art. 42 become zero except that one containing the single P under consideration. When the single load is in the span l_2 (or l) the terms in which are found $\overset{1}{\Sigma}$ and $\overset{2}{\Sigma}$ become zero, and similarly for a single load in each of the other spans. In illustration of this part of the analysis, if a single load P rests on the main span l_2:

$$M_3 = -l_2 \frac{(2a + r) P (1 - q^2) q}{4 (a + r) (br' + 1) - r} \quad . \quad . \quad . \quad 176.$$

$$M_2 = -l_2 \frac{r (2br' + 1) P (1 - q^2) q}{4 (a + r) (br' + 1) - r} \quad . \quad . \quad 177.$$

For a single load on span l_3:

$$M_3 = -l_2 \frac{2b^2 (a + r) r' P (1 - m^2) m}{4 (a + r) (br' + 1) - r} \quad . \quad . \quad 178.$$

$$M_2 = l_2 \frac{b^2 r r' P (1 - m^2) m}{4 (a + r) (br' + 1) - r} \quad . \quad . \quad . \quad 179.$$

For a single load on span l_1:

$$M_3 = l_2 \frac{a^2 P (1 - n^2) n}{4(a+r)(br^1+1) - r} \quad . \quad . \quad . \quad . \quad . \quad 180.$$

$$M_2 = -l_2 \frac{2a^2(br^1 + 1) P (1 - n^2) n}{4 (a + r) (br' + 1) - r} \quad . \quad . \quad 181.$$

If, as is usual, $l_1 = l_3$, *i.e.*, $a = b$, and $r = \frac{1}{r'}$, the preceding equations become:

For single load P in the main span l_2:

$$M_2 = -l_2 \frac{P(1-q^2)q}{2\frac{a}{r}+3} \quad . \quad . \quad . \quad . \quad 182.$$

$$M_2 = -l_1 \frac{P(1-q^2)q}{2\frac{a}{r}+3} = M_3 \quad . \quad . \quad . \quad 183.$$

For a single load on span l_3:

$$M_3 = -l_2 \frac{a\left(1+\frac{r}{a}\right)P(1-m^2)m}{2+4\frac{r}{a}+\frac{3}{2}\frac{r^2}{a^2}} \quad . \quad . \quad 184.$$

$$M_2 = l_2 \frac{rP(1-m^2)m}{4+8\frac{r}{a}+3\frac{r^2}{a^2}} \quad . \quad . \quad . \quad . \quad 185.$$

For a single load on span l_1:

$$M_3 = l_2 \frac{rP(1-n^2)n}{4+8\frac{r}{a}+3\frac{r^2}{a^2}} \quad . \quad . \quad . \quad . \quad 186.$$

$$M_2 = -l_2 \frac{a\left(1+\frac{r}{a}\right)P(1-n^2)n}{2+4\frac{r}{a}+\frac{3}{2}\frac{r^2}{a^2}} \quad . \quad . \quad 187.$$

If H' is the horizontal cable stress due to moving load and s' the corresponding uniform suspender pull per unit of span, while f is the sag of the cable carrying the load, there may at once be written:

$$\frac{s'l_2^2}{8} = H'f \text{ and } \frac{s'l_2^2}{4} = 2H'f \quad . \quad . \quad . \quad 188.$$

Eq. 19 of Art. 42 will then become:

$$M'' = 2\,H'\,f\,B' \quad . \quad . \quad . \quad . \quad . \quad 189.$$

In this equation:

$$B' = \frac{a^3 - 2\,a - r - 2\,b^3\,(a+r)\,r'}{4\,(a+r)\,(b\,r'+1) - r} = \frac{4\,B}{l_2^2} \quad . \quad . \quad 190.$$

Similarly, Eq. 20 of Art. 42 becomes:

$$M' = -\,H'\,f\,C' \quad . \quad . \quad . \quad . \quad . \quad 191.$$

C' has the value:

$$C' = \frac{r}{a+r}\left(\frac{a^3-2\,a-r-2\,b^3\,(a+r)\,r'}{4\,(a+r)\,(br'+1)-r} + \frac{a^2}{r} + 1\right) = \frac{8\,C}{l_2^2} \quad . \quad 192.$$

In the usual case of the side spans being alike, $l_1 = l_2$, $a = b$, and $r = \frac{1}{r'}$. Hence:

$$B' = -\,\frac{a^2+r}{2\,a+3\,r} \quad . \quad . \quad . \quad . \quad . \quad 193.$$

$$C' = 2\,\frac{a^2+r}{2\,a+3\,r} \quad . \quad . \quad . \quad . \quad . \quad 194.$$

Hence, for this case:

$$M'' = M' = -\,2\,H'\,f\,\frac{a^2+r}{2\,a+3\,r} \quad . \quad . \quad . \quad 195.$$

Remembering that when z is less than x, $(z \leq x)$, $M_0 = P\,z\left(1 - \frac{x}{l}\right)$; and that when z is greater than x, $(z \geq x)$, $M_0 = P\left(1 - \frac{z}{l}\right)x$, Eqs. 82, 83, and 84 of Art. 47 will become:

For span l_1:

$$M = Pz\left(1 - \frac{x}{l_1}\right)_{z \leq x} \text{ or } P\left(1 - \frac{z}{l_1}\right)_{z \geq x} x + M_2 + \frac{x}{l_1}$$

$$+ \left(C'\,\frac{x}{l_1} - \frac{4}{l_2^2}\,(l_1 - x)\,x\right) f\,H' \quad . \quad . \quad . \quad . \quad 196.$$

For span l_2:

$$M = P z \left(1 - \frac{x}{l_2}\right)_{z \leq x} \text{ or } P \left(1 - \frac{z}{l_2}\right)_{z \geq x} x + M_2 + (M_3 - M_2)\frac{x}{l_2}$$

$$+ \left(C' - C'\frac{x}{l_2} - 2B'\frac{x}{l_2} - \frac{4}{l_2^2}(l_2 - x)\,x\right) f H' \quad . \quad . \quad 197.$$

For span l_3:

$$M = P z \left(1 - \frac{x}{l_3}\right)_{z \leq x} \text{ or } P \left(1 - \frac{z}{l_3}\right)_{z \geq x} x + M_3 \left(1 - \frac{x}{l_3}\right)$$

$$- \left(\left(1 - \frac{x}{l_3}\right) 2B' + \frac{4}{l_2^2}(l_3 - x)\,x\right) f H' \quad . \quad . \quad 198.$$

As the preceding equations are written, there may be a load P on each of the three spans concurrently, or on any two, or on one only. The first term of each second member will be zero for each span which carries no load P.

Eqs. 196, 197, and 198 are equations of the influence lines for moments in the three spans. Under the usual procedures some arbitrary fixed or constant value as unity would be assigned to P. Then if that unit load be placed on any span while the other two spans are free of such load, and if x should remain unchanged in each span while z takes all values desired in the span where it is found, values of the bending moment M in each of the three spans for the unit load will result. Such moments can then be laid off by any convenient scale from the points of application of the loading and from corresponding points in the other two spans. The lines so constructed will be the moment influence lines for all the spans. The actual moments will be found by multiplying the unit moments by the actual loads expressed in terms of the unit loads. The moments M_2 and M_3 must be computed for the same unit loads by Eqs. 176 to 187, while H' for the same unit (moving) load is also to be found for each point of its application by Eq. 36, Art. 43, in the general case, after making all terms zero in which the dead load appears, or by other equations adapted from Eq. 36 to the conditions of special cases.

Usually $l_1 = l_3$ and $r = \frac{1}{r'}$, so that $a = b$, and the simplified forms of B' and C' are to be employed as indicated by Eqs. 193 and 194.

Shears.

The shears at any or all points in the three spans can at once be written from the moments, Eqs. 196, 197, and 198, by simply taking the values of $\frac{d\,M}{d\,x}$. This operation will give:

Shear for span l_1:

$$S = \left(P\frac{z}{l_1}\right)_{z \leq x} \text{ or } P\left(1 - \frac{z}{l_1}\right)_{z \geq x} + \frac{M_2}{l_1} + \left(\frac{C'}{l_1} - \frac{4}{l_2^2}(l_1 - 2x)\right) f H'. \quad 199.$$

Shear for span l_2:

$$S = \left(P\frac{z}{l_2}\right)_{z \leq x} \text{ or } P\left(1 - \frac{z}{l_2}\right)_{z \geq x} + \frac{M_3 - M_2}{l_2} - \left(\frac{C'}{l_2} + \frac{2\,B'}{l_2} + \frac{4}{l_2^2}(l_2 - 2x)\right) f H' \quad . \quad . \quad . \quad 200.$$

Shear for span l_3:

$$S = \left(P\frac{z}{l_3}\right)_{z \leq x} \text{ or } P\left(1 - \frac{z}{l_3}\right)_{z \geq x} - \frac{M_3}{l_3} + \left(\frac{2\,B'}{l_3} - \frac{4}{l_2^2}(l_3 - 2\,x)\right) f H' \quad . \quad . \quad . \quad . \quad 201.$$

These three equations are equations of the influence lines for shear in all the spans. As for the moment lines, a unit value is to be assigned to P usually taken in any one span only. The values found for M_2, M_3, and H' for the moment influence lines are available for the shear lines also.

The influence lines for shears may thus be completely constructed for all cases of continuous trusses.

Art. 58.—Influence Lines for Moments and Shears—Non-Continuous Trusses.

Moments.

The equations of influence lines for moments in non-continuous stiffening-trusses may be written directly from the equations of the preceding article. As in this case, $M_2 = M_3 = M' = M'' = 0$, the bending moment in the stiffening-truss is:

$$M = \overset{z \leq x}{Pz} \text{ or } \overset{z \geq x}{Px} - P\,q\,x - \frac{4 f_2 H'}{l_2^2}(l_2 - x)\,x \quad . \quad 202.$$

The arrangement of this equation indicates that when z is less than x, $P\,z$ is to be used in the first term of the second member, but that $P\,x$ is to be used when z is greater than x.

The same equation is applicable to either side span by simply changing the subscript 2 to 1 or 3.

The general value of H' for a single load in each span is given by Eq. 123 of Art. 52, after substitution for the integrals which it contains from Eqs. 71, 72, and 73 of Art. 46:

$$H' = \frac{\frac{F_0}{I}\left\{\frac{f_2 l_2^2}{3} P\,q\,(1 - q^2(2-q)) + \frac{1}{r}\frac{f_1 l_1^2}{3} P\,q\,(1 - q^2(2-q))\right.}{D}$$

$$\left. + \frac{1}{r}\frac{f_3 l_3^2}{3} P\,q\,(1 - q^2(2-q))\right\}^{*} \quad . \quad . \quad 203.$$

Usually the influence line is constructed for a single load P in one span only. In that case all the terms in the numerator of the second member of Eq. 203, except that containing the load P in the one span considered, become zero. The load P will ordinarily represent a panel load.

If the single load P is in the span l_2 only, Eq. 202 will take the form for that span, by the aid of Eq. 203:

$$M = \overset{z \leq x}{Pz} \text{ or } \overset{z \geq x}{Px} - Pqx - \frac{4 f_2^2 F_0}{3 I D} Pq\,(1 - q^2(2-q))\,(l_2 - x)\,x \quad . \quad 204.$$

* This quantity can quickly be computed by the aid of the table of numerical values at the end of this chapter.

In this equation $P z - P q x$ and $P x - P q x$ are always positive, and so is $1 - q_2 (2 - q)$.

This Eq. 204 is the equation of the influence line for moments in the centre or main span caused by a single (panel) load in that span. If x is given a fixed value with P, taken as unity while z or q is made to vary throughout the span, M will be the unit bending moment at the point located by x caused by a unit load at z, *i.e.*, it will be the ordinate of the influence line to be drawn at the point z.

If any one of the three spans, as l_1, carries no load P, the moment for any point of that span will be:

$$M = - \frac{4 f_1 H'}{l_1^2} (l_1 - x) x = - \frac{s'}{2} (l_1 - x) x \quad . \quad . \quad 205.$$

This negative moment is due wholly to the upward moving load pull of the suspenders. The negative reaction, *i.e.*, the anchorage required at the end of such a span, is evidently $\frac{s' l}{2}$.

Eq. 205, giving the value of M, is the equation of a parabola with the origin of co-ordinates at the end of the span and with the vertex of the parabola on a vertical line through the centre of the span. The ordinates of such a parabola represent, therefore, the bending moments in a non-continuous span, carrying the uniform load s', at those points of the span from which they are drawn.

In such a case $P = 0$ in the terms $P z$ or $P x - P z \frac{x}{l}$ in the second member of Eq. 202. The horizontal stress H' will be given by Eq. 203 after making $P = 0$ for all spans except that one in which P is found. If the load P is in either span l_1 or l_3, the factor $\frac{F_0}{I}$ in the value of M, Eq. 204 must be changed to $\frac{F_0}{rI}$.

It is convenient in constructing influence lines to put Eq. 202

in a different form. When $z \leq x$ the first two terms of the second member may be written $\frac{Pz}{l_2}(l_2 - x)$, hence:

$$M = \left(P\frac{z\,l_2}{4\,f_2\,x} - H'\right)\frac{4\,f_2}{l_2^2}(l_2 - x)\,x \quad . \quad . \quad . \quad 206.$$

When $z \geq x$ the first two terms of the second member become $\frac{Px}{l_2}(l_2 - z)$, hence:

$$M = \left(P\,\frac{l_2\,(l_2 - z)}{4\,f_2\,(l_2 - x)} - H'\right)\frac{4\,f_2}{l_2^2}(l_2 - x)\,x \quad . \quad . \quad 207.$$

Obviously the first term of the first parenthesis in the second member of the two equations is the ordinate of a straight line. If $z = x$, whatever the value of the latter may be, that ordinate is $P\,\frac{l_2}{4\,f_2}$; if $z = 0$, or $z = l_2$, that ordinate is zero. These terms are therefore ordinates of two straight lines drawn from each end of the span to the end of the common ordinate $P\,\frac{l_2}{4\,f_2}$ erected at the point z. The moment ordinate M may then be represented by the difference between any such ordinate and H'.

If it be supposed that a single load P rests in the span l_2 only, remembering that $\frac{f_1}{l_1^2} = \frac{f_2}{l_2^2}$, Eq. 202 will then give the bending moment M for the span l_1, as will Eq. 205 by the aid of Eq. 203:

$$M = -\frac{4\,f_2^2}{3}\frac{F_0}{I\,D}P\,q\,(1 - q^2\,(2 - q))\,(l_1 - x)\,x \quad . \quad 208.$$

The value of q (position of load P) in the span l_2 for the greatest value of H' and the greatest bending moment in span l_1 is found by the equation:

$$\frac{d\,M}{d\,q} = \frac{d\,H'}{d\,q} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad 209.$$

By making the differentiation indicated:

$$1 - 6q^2 + 4q^3 = (1 - 2q)(1 + 2q - 2q^2) = 0;$$

Or:

$$1 - 2q = 0 \therefore q = \tfrac{1}{2} \quad . \quad . \quad . \quad . \quad 210.$$

This result, however, is essentially obvious without analytic demonstration.

If the span l_2 were traversed by a load of varying density, the position of the point of greatest density for the maximum bending moment in the span l_1 would be given by the above value of q. The same value of q evidently gives the greatest value of the horizontal stress H' for all three spans loaded concurrently.

No Stiffening-Trusses in Side Spans.

If there is a stiffening-truss in the span l_2 only, the bending moment M will be given by Eq. 204 as it stands, while Eq. 203 will give H' after making $P = 0$ in spans l_1 and l_3.

With no load in spans l_1 and l_3, but with a load P in span l_2, Eq. 203 will give the following value already used:

$$H' = \frac{F_0}{DI}\left\{\frac{f_2 l_2^2}{3} P q (1 - q^2 (2 - q))^*\right\} \quad . \quad . \quad 211.$$

The general Eq. 202, or more conveniently its special form Eq. 204, with $P = 1$, is the equation of the influence line for bending moments in the stiffening-truss for the case of such a truss in the main span only. By taking a fixed value of x and then varying q or z throughout the span the influence line for any point x of the span may readily be drawn.

Unit Line Showing the Horizontal Stress H'.

By taking $P = 1$, in Eqs. 203 or 211, and then varying z or q throughout the span, unit values of H' may be computed and laid

* See Table of numerical values at the end of this chapter.

off as ordinates of a curve representing the value of H' for each position of the unit load in the span. Those unit ordinates multiplied by any load P will give the horizontal stress H' produced by P placed at any point indicated by z or q, whether moving load rests on any one span or on more than one span. The curved line will be symmetrical about a vertical through the centre of the span.

Shears.

The shear S for any span carrying a load P is given directly by Eq. 202, by simple differentiation as follows: first when z is less than x:

$$S = \frac{dM}{dx} = -Pq - \frac{4f_2H'}{l_2}\left(1 - \frac{2x}{l_2}\right) \quad . \quad . \quad 212.$$

When z is greater than x:

$$S = \frac{dM}{dx} = P(1-q) - \frac{4f_2H'}{l_2}\left(1 - \frac{2x}{l_2}\right) \quad . \quad . \quad 213.$$

For a span carrying no moving load, Eq. 205 is to be used, and the shear S will take the value:

$$S = \frac{dM}{dx} = -\frac{4f_1H'}{l_1}\left(1 - \frac{2x}{l_1}\right) = \frac{s'l_1}{2}\left(1 - \frac{2x}{l_1}\right) \quad . \quad 214.$$

These three equations are equations of influence lines for shears if $P = 1$ and if x be taken as fixed while z or q varies throughout the length of the span, and those lines may readily be constructed, taking the value of H from Eq. 203 after making $P = 0$ for each of those spans carrying no moving load.

It should be observed that if x represents the half span in Eqs. 212, 213, and 214, the term in which it appears reduces to zero. In other words, the shear at the centre of the span of a non-continuous stiffening-truss is precisely the same as the shear for a non-continuous simply supported truss of the same span and carrying the same load as the stiffening-truss, whatever may be the position of the loading.

Influence Lines for Deflection.

By referring to the formulæ for deflection developed in preceding articles both for continuous and non-continuous stiffening-trusses, it will be found that by making the load P equal to unity and giving to x a fixed value, while z or q varies throughout the span, influence lines may readily be constructed for deflections in precisely the same manner as for moments and shears. In fact, the operations are so nearly identical with those of moments and shears that it is not necessary to reproduce here the equations of the influence lines for deflections. It may be stated in general that influence lines may be written in precisely the same manner for any quantities which vary directly with the imposed load.

Art. 59.—Application of Influence Line Equations to an Actual Bridge.

A few applications of the preceding influence line formulæ will be made, to illustrate their use, by the aid of data belonging to the Manhattan wire cable stiffened suspension bridge across the East River at New York City, the outline of which, with main dimensions of span and height of tower, is shown on Plates I and II. This bridge was designed by Mr. L. S. Moiseieff, C.E., by the aid of formulæ which he developed from the fundamental equations given by Melan. The analysis of Mr. Moiseieff used in his design is based upon the method of treatment by deflections, which will be set forth in the next chapter.

The Manhattan Bridge consists of three spans as shown, the main span being 1,470 feet in length between centres of towers, while each of the end spans is 725 feet in length from the centre of the tower to the centre of the end pin at the other end of the span. The centre sag of the cable in the main span, *i.e.*, the height from the point in the cable over the end pin of the stiffening-truss to the lowest point of the cable is 145.3 feet, while the centre sag of each side cable below the mid-point of the inclined chord drawn from the point in the cable over the centre of the end pin

of the stiffening-truss adjacent to the tower to a point in the cable over the centre of the other end pin is 37.2 feet. The height of the tower above the point of support of the lower chord of the stiffening-truss is 180 feet, but the total height of the steel tower above the masonry supporting it is 281.9 feet.

The trusses are non-continuous, also as shown on Plates I and II. The centres of end pins of the trusses at the feet of the towers at the extremities of the main span are separated by 23.15 feet. Inasmuch as there are stiffening-trusses in each of the three spans, the formulæ to be used for the horizontal component of cable stress, bending moments, and deflections are those given in Arts. 46, 52, 53, and 55.

Before making the actual computations it is necessary to observe that the actual span lengths to be used are not quite equal to the lengths measured from centre to centre of the towers. The length of span to be taken is that of the stiffening-truss in which moments, shears, and other stresses, as well as deflections, are to be computed, although this may not be strictly accurate for the cable. In these computations the length of main span to be employed is 1,446.7 feet and 713.5 feet for each end span. In this bridge there are four cables carrying four stiffening-trusses.

H′ and Moments.

The values of the bending moment to be used here are Eqs. 202 to 208 of Art. 59, which require the value of H' given in the general case by Eq. 222. The data to be employed are as follows:

$$l_2 = 1446.7 \text{ ft.} \quad l_1 = 713.5 \text{ ft.} \quad \frac{f_2}{l_2} = .1$$

$$f_2 = 145.3 \text{ ft.} \quad f_1 = 37.2 \text{ ft.} \quad \frac{f_1}{l_1} = .0522$$

$$\tan\alpha + \tan\alpha_1 = .8617; \; h' = 282 \text{ ft.}, \; h'' = 176 \text{ ft.}$$

As the foot is taken as the unit of length the moment of inertia I_2 of the sectional area of the chords of the truss of the centre span should be taken in the same unit as well as the sectional areas themselves. Inasmuch, however, as I_2 appears in the formula for H' only in the ratio $\frac{F_0}{I_2}$, and as the sectional

areas appear also in the ratios $\frac{F_0}{F_2}$ and $\frac{F_0}{F_3}$, those areas may be expressed in square inches. If I_2 be considered as the product of the sectional areas of the chords of the truss expressed in square inches multiplied by the square of the radius of gyration expressed in feet, the value of the ratio $\frac{F_0}{I_2}$ will be the same as if the foot were the unit both for the sectional area and the radius of gyration. Expressed in this manner, then, per truss:

$I_2 = 44{,}000$ (sq. ins. $\times \overline{ft^2}$). $r = \frac{I_2}{I_1} = 1.16$.

$I_1 = 51{,}000$ (sq. ins. $\times \overline{ft^2}$).

$F_0 = 275$ sq. ins. for one cable.

$F_2 = .1$ sq. in. per linear ft. of span per cable.

$F_3 =$ (average value per cable) $= 400$ sq. ins.

Unit Load in Span l_2.

Eq. 203 gives the value of H' desired, the denominator D having the value shown in Eq. 70 of Art. 46. If a single load P rests on the span l_2, the other spans being free of load, Eq. 203 will take the form:

$$H' = \frac{F_0 f_2 l_2^2}{3 I_2 D} P q (1 - q^2 (2 - q)) \quad . \quad . \quad . \quad 215.$$

By the aid of the table at the end of this chapter Eq. 215 gives the following tabulation after making the load $P = 1$.

q	$5.731\, q (1 - q^2 (2 - q)) = H'$
.05	.285,2
.1	.562,2
.15	.823,9
.2	1.063,7
.25	1.276,1
.3	1.456,2
.35	1.600,4
.4	1.705,5
.45	1.769,5
.5	1.791

By making the numerical substitutions in the denominator of the second member of Eq. 70, Art. 46, it will be found that

$$D = 110{,}710; \quad \text{and hence,} \quad \frac{F_0 f_2 l_2^2}{3 I_2 D} = 5.731 \quad . \quad 216.$$

If $P = 1$, *i.e.*, if a unit load be assumed, the second column in the above tabulation will give the values of H' for the unit load at the points indicated by q, *i.e.*, by z. It is clear that these values need be computed for one-half the span only, as those in the other half will be symmetrically the same. The value of H' for any other load than unity will be found by simply multiplying the preceding tabular value for the same q or z by P.

The bending moments on which are based the influence lines for moments at the quarter and central points of the main span (or at any other points) may now readily be computed. If $x = \frac{l_2}{4}$ in Eqs. 206 and 207, they become:

When $z \leq x$:

$$M = \left(P \frac{z}{f_2} - H'\right) \frac{3 f_2}{4} \quad . \quad . \quad . \quad . \quad 217.$$

When $z \geq x$:

$$M = \left(P \frac{l_2 - z}{3 f_2} - H'\right) \frac{3 f_2}{4} \quad . \quad . \quad . \quad . \quad 218.$$

Again, if $x = \frac{l_2}{2}$, the same two equations become:

When $z \leq x$:

$$M = \left(P \frac{z}{2 f_2} - H'\right) f_2 \quad . \quad . \quad . \quad . \quad . \quad 219.$$

When $z \geq x$:

$$M = \left(P \frac{l_2 - z}{2 f_2} - H'\right) f_2 \quad . \quad . \quad . \quad . \quad 220.$$

The numerical values of H' for varying values of z are to be taken from the tabulation following Eq. 215. The co-ordinate z will be taken in tenths of the span from $z = 0$ to $z = l_2$.

As unit loads, P, are to be taken for influence-line purposes, $P = 1$ in Eqs. 217 to 220.

After making the numerical substitutions in the preceding four equations the following tabulated values will be found, the second column by the aid of Eqs. 217 and 218, and the third column by the aid of Eqs. 219 and 220. The values of z are so taken as to make $\frac{z}{l_2}$, .1, .2, .25, .3, .4, etc.

z	M for $x = \frac{l_1}{4}$	M for $x = \frac{l_2}{2}$
144.67 ft.	+ 47.718 ft. lbs.	− 9.038 ft. lbs.
289.34 "	+ 102.038 "	− 9.256 "
361.675 "	+ 133.375 "	.....
434.01 "	+ 95.571 "	+ 6.364 "
578.68 "	+ 32.082 "	+ 42.791 "
723.35 "	− 13.946 "	+ 103.018 "
868.02 "	− 40.571 "	+ 42.791 "
1012.69 "	− 49.725 "	+ 6.364 "
1157.36 "	− 43.274 "	− 9.256 "
1302.03 "	− 24.944 "	− 9.038 "

These tabular values will be used in the graphical representation of the influence lines for moments at the two points indicated.

The bending moments in the unloaded spans, l_1 and l_3 (identical, as the two spans are equal), are found by Eq. 205 after giving H' the successive values shown in the tabulation following Eq. 215, each value of q having a corresponding value of H'. The greatest negative bending moments evidently exist when $q = .5$, *i.e.*, when the unit load is at the centre of the main span, l_2, as indicated by Eq. 210. The parabolic curve for this position of the unit load, only, will be constructed. The corresponding value of H' is 1.791. Hence, using the data already given:

$$\frac{4 f_1 H'}{l_1^2} = .000,524,1$$

For this case, therefore, Eq. 205 becomes:

$$M = -.000,524,1\, l_1^2 \left(1 - \frac{x}{l_1}\right)\frac{x}{l_1} = -266.81 \left(1 - \frac{x}{l_1}\right)\frac{x}{l_1} \quad . \quad 221.$$

If $\frac{x}{l_1}$ take the successive values, .15, .3, .4, and .5, there will result:

$$\frac{x}{l_1} = \quad .15 \quad .3 \quad .4 \quad .5$$

$$M = -34.02;\ -56.03;\ -64.03;\ -66.7 \quad . \quad 222.$$

Evidently the moment M is zero at each end of the span, *i.e.*, when $\frac{x}{l_1}$ is 0 or 1.

On Plate II the skeleton of the structure and the dimensions required for the construction of the two influence lines are given with sufficient clearness to require no further explanation.

The curved line $B_2 C'_2 D_2$ is constructed from the second column in the tabulation following Eq. 215, so as to make the vertical ordinate drawn from any point of the horizontal line $B_2 D_2$ downward to the curve represent the value of H' for a unit load placed at that point from which the ordinate is drawn, *i.e.*, the ordinate $C_2 C'_2$ represents 1.791, while $b G_2$ represents 1.769,5. These ordinates are all drawn full size.

The horizontal line $B'_2 D'_2$ is drawn at the distance $C_2 F_2$ below $B_2 D_2$, represented by $\frac{l_2}{4 f_2} = 2.49$ (drawn full size), as shown by the paragraph following Eq. 207. If straight lines be drawn from any point of $B'_2 D'_2$, as E_2, to each extremity of $B_2 D_2$, the difference between any vertical ordinate as $J_2 J'_2$ drawn to either one of those straight lines and the ordinate from the same point J_2 to the H' curve, as $J_2 J''_2$, will represent the bending moment at the point in the stiffening-truss over the point of intersection of the two straight lines, produced by a unit load placed at the point vertically over J_2 from which the vertical ordinate is drawn. Hence $J''_2 J'_2$ will represent the bending moment at the point E in the truss produced by a unit load placed at the point J, and it will have a positive sign because the ordinate to the H' curve is smaller at that point than the ordinate to the straight line. Similarly $O_2 O'_2$ represents a negative moment

produced at the quarter point of the span by a unit load placed in the stiffening-truss at N. This moment is negative because $N_2 O'_2$ (the ordinate to the H' curve) is greater than $N_2 O_2$, the ordinate to the straight line $E_2 D_2$. In general, the difference between vertical ordinates drawn to any pair of straight lines intersecting in the horizontal line $B'_2 D'_2$ and to the H' curve will represent ordinates to the moment influence line for that point in the stiffening-truss vertically over the point of intersection of those straight lines. These moment ordinates, however, must be multiplied by the quantity:

$$\frac{4 f_2}{l_2^2} (l_2 - x) x = 4 f_2 \left(1 - \frac{x}{l_2}\right) \frac{x}{l_2}$$

for the span l_2, and by a similar quantity for either of the spans l_1 or l_3 in order to make them equal the unit moment as shown by Eqs. 206 and 207. Or, what amounts to the same thing, the above quantity may be considered the scale at which the ordinates described are drawn, *i.e.*, $\frac{3 f_2}{4} = 109$ for $x = \frac{l_2}{4}$ and $f_2 = 145.3$ for $x = \frac{l_2}{2}$.

Although this is the frequent and perhaps usual procedure to determine moment influence lines, a clearer and in some respects more satisfactory method is to compute the actual unit moments for a series of points throughout the span, as has been done in the tabulation following Eq. 220 by the aid of Eqs. 217 to 220. As indicated in connection with that tabulation, points one-tenth of the span apart have been taken for illustration, although sections of the stiffening-truss closer together would be required in an actual computation for such a span length as that under consideration. The two curves $B_2 M_2 D_2$ and $B_2 M'_2 D_2$, Plate I, are the moment influence lines for the one-fourth and centre points of the span, their vertical ordinates being given by the second and third columns in the tabulation following Eq. 220.

Similar influence lines may be drawn for every other tenth point, or other desired points in the span.

Influence lines drawn in this manner are of much value in determining at a glance the portion of the span to be covered by the moving load in order to produce the greatest positive or negative moment at any point.

The influence line $B_2 M_2 D_2$ shows that all loads between B_2 and b will produce positive moments, *i.e.*, those causing downward deflection, and that all loads between D_2 and b will produce negative moments at the left quarter point of the span. Hence, to cause the greatest positive bending moment at that quarter point, the part $B_2 b$ of the span must be covered with moving load, while $b D_2$ and the two side spans must carry none. On the other hand, the part $D_2 b$ of the span must be covered with moving load while none rests on $B_2 b$ for the greatest negative moment at the same quarter point, both side spans being fully loaded at the same time, as such loading causes negative moments in the centre span.

Similarly, the influence line for the centre point of the truss shows that the greatest positive moment at that point will be found by loading the segment $b' b'$ only of the span (the side spans being unloaded), and that the concurrent loading of the entire and equal segments $B_2 b'$ and $D_2 b'$ only will cause the greatest centre negative moment, but again with both side spans fully loaded.

The same results are obviously shown by the intersections of the straight lines drawn from E_2 and F_2 with the H' curve, Plate II.

If p be the length of a panel of the stiffening-truss, the panel load will be p multiplied by unity and the corresponding moment will be p multiplied by the ordinate drawn to an influence line as $B_2 M_2 D_2$, Plate I. The summation of the quantities p multiplied by moment ordinates for all those points required to give the greatest moment at the section under consideration will obviously give the greatest unit load moment for that section. The moment ordinate of the influence line for a unit load at any point of the stiffening-truss multiplied by the panel length is approximately a part of the area included between the influence line and the horizontal or span line $B_2 D_2$. This approximation is exact if

the panel length be supposed indefinitely short, *i.e.*, $d\,x$. Hence, the area $B_2\,M_2\,b$, Plate I, expresses the greatest positive moment at the quarter point of the span produced by a uniform load of one unit (as one pound or one ton) per unit of span extending from B_2 to b. Similarly the area $b\,M_2\,D_2$ will represent the greatest negative bending moment at the quarter point of the span caused by a uniform load of one unit per unit of span extending from b to D_2. If one pound, or one ton, be the unit of load and the foot the unit of length, each area indicated will represent a moment of one foot pound or one foot ton, as the case may be, for each square foot, if the vertical scale for the ordinate of the influence line be one foot pound or one foot ton per vertical foot. If the area so determined be multiplied by the actual uniform load per linear unit of span the result will be the greatest bending moment for the actual uniform load to which the stiffening-truss may be subjected.

Evidently the areas included between other influence lines and the horizontal line $B_2\,D_2$ bear the same interpretation. In any case the difference between the positive and negative areas will represent the bending moment at the section of the stiffening-truss under consideration produced by a uniform unit load for the entire span.

The negative moments in the span l_1 and l_3, caused by any single load, placed in the span l_2 may be represented by the parabolic moment curve $A_1\,m_1\,B_1$, Plate I, constructed from the values given by Eq. 222 and computed from Eq. 221. The vertical ordinates are laid off to a scale of 50 moment units, as foot pounds, per linear inch, the centre ordinate $m_1\,C_1$ representing the negative moment—66.7.

As explained in connection with Eq. 221, this moment curve represents the maximum negative moments in the span l_1 produced by a unit load at the centre of the span l_2. A similar parabolic moment curve is required to represent the negative bending moments caused in the span l_1 by a unit load placed at any other point than the centre of the span l_2. Furthermore, Eq. 205 shows that the negative bending moment in an unloaded span varies directly as H', while Eq. 203 shows that the latter

increases with the load on a loaded span. The resultant negative bending moment, therefore, at any point in the unloaded span l_1 or l_3 will be found by taking the sum of the negative moments produced by the concurrent action of all possible loads on the spans l_2 and l_3 or l_1. Evidently the same parabolic negative moment curves are required for the unloaded span l_3 as for l_1.

Unit Load in Side Span l_1 or l_3.

If a unit load rests on the span l_1, the bending moments are given by Eqs. 206 and 207 after changing the subscript 2 to 1. The value of H' is given by Eq. 203:

$$H' = \frac{F_0 f_1 l_1^2}{3 I_1 D} q (1 - q^2 (2 - q)) \quad . \quad . \quad . \quad 223.$$

When H' is found, Eq. 205 will give the negative moments in the unloaded spans l_2 and l_3.

Using the numerical values of f_1, l_1, I_1, and F_0 already given and taking the computed value of D (110,710), there will result:

$$H' = .307,46\, q (1 - q_2 (2 - q)) \quad . \quad . \quad . \quad 224.$$

Hence, by the aid of the second column in the tabulation following Eq. 215 there will be found:

q	$.307,46\, q (1-q^2 (2-q)) = H'$ for unit load.
.05	.015,31
.1	.030,17
.15	.044,21
.2	.057,07
.25	.068,47
.3	.078,14
.35	.085,87
.4	.091,51
.45	.094,94
.5	.096,09

If again the influence lines for the points $x = \frac{l_1}{4}$ and $x = \frac{l_1}{2}$ be constructed, Eqs. 217, 218, 219, and 220 may be employed

as they stand by simply changing the subscript from 2 to 1, and making $P = 1$. Proceeding in this manner and taking the values of H' from the tabulation above, the following bending moments at the quarter and centre points for a unit load placed successively at the locations shown by z or q will be found.

z	M for $x = \frac{l_1}{4}$	M for $x = \frac{l_1}{2}$
71.35 ft.	+ 52.68	+ 34.52
142.7 "	+ 105.43	+ 69.15
178.375 "	+ 129.09	.....
214.05 "	+ 122.54	+ 104.01
285.4 "	+ 104.32	+ 139.13
356.75 "	+ 86.41	+ 174.59
428.1 "	+ 68.69	+ 139.13
499.45 "	+ 51.25	+ 104.01
570.8 "	+ 34.04	+ 69.15
642.15 "	+ 16.97	+ 34.52

The H' curve is laid down on Plate II in the span l_1 from the tabulation following Eq. 224 (and computed by that equation), as $A_1 C'_1 B_1$, the vertical ordinates being laid down at full size as shown in the tabulation. The horizontal line $A'_1 B'_1$ is laid off as shown on the Plate at the distance $\frac{l_1}{4 f_1} = 4.8$, as required by Eqs. 206 and 207. E_1 and F_1 are placed at the quarter and centre points of the span. Inasmuch as the straight lines drawn from E_1 and F_1 to A_1 and B_1 do not cross the H' curve, all moments at the quarter and centre points produced by unit loads will be positive.

The actual bending moments produced at the quarter and centre points of the span by unit loads placed successively at points one-tenth of the span length apart are given in the second tabulation following Eq. 224, and they are shown laid down on Plate I as the two influence lines, $A_1 M_1 B_1$ for the quarter point and $A_1 M' B_1$ for the centre point, the scale being 100 moment units per inch.

In the unloaded span l_2 the negative bending moments are

readily computed by Eq. 205, which takes the form, remembering that $f_3 = 37.2$ feet:

$$M = 4 f_3 H' \left(1 - \frac{x}{l_3}\right) \frac{x}{l_3} = 148.8 H' \left(1 - \frac{x}{l_3}\right) \frac{x}{l_3} \quad . \quad 225.$$

Taking the unit load in the centre of span l_1 for which $H' = .096{,}09$:

$$\frac{x}{l_3} = .15 \qquad .3 \qquad .4 \qquad .5$$

$$M = .838{,}75; \quad 2.441{,}7; \quad 3.268; \quad 3.574{,}5$$

By plotting these values the moment curve $D_3 m_3 K_3$, Plate I, results, which shows at 10 moment units per inch the greatest negative moments in the span l_3 due to any single unit load in span l_1, as the unit load is at the centre of the latter span. The entire span l_1 must be loaded to give maximum negative moments at all points in the span l_3 due to loading in l_1.

Similarly for the unloaded span l_2, remembering that $f_2 = 145.3$ feet and that $H' = .096{,}09$, *i.e.*, for a unit load at the centre of span:

$$\frac{x}{l_2} = .15 \qquad .3 \qquad .4 \qquad .5$$

$$M = 3.276{,}2; \quad 9.537{,}3; \quad 12.764{,}8; \quad 13.962$$

The moment curve $B_2 m_2 D_2$, Plate I, drawn at 10 moment units per inch of vertical ordinate, shows the negative moments for the entire span. If H' be taken for the entire span l_1 covered with moving load, the resulting negative moments will be the greatest possible for moving load on that span (l_1) only.

These influence lines for moments and the H' curves illustrate fully the procedures to be followed to determine the positions of loading for the greatest moments in all three spans, and the corresponding values of H'.

The greatest negative bending moments in any one span will be found when the other two spans are loaded over their entire lengths, with the amounts and positions of moving load on the

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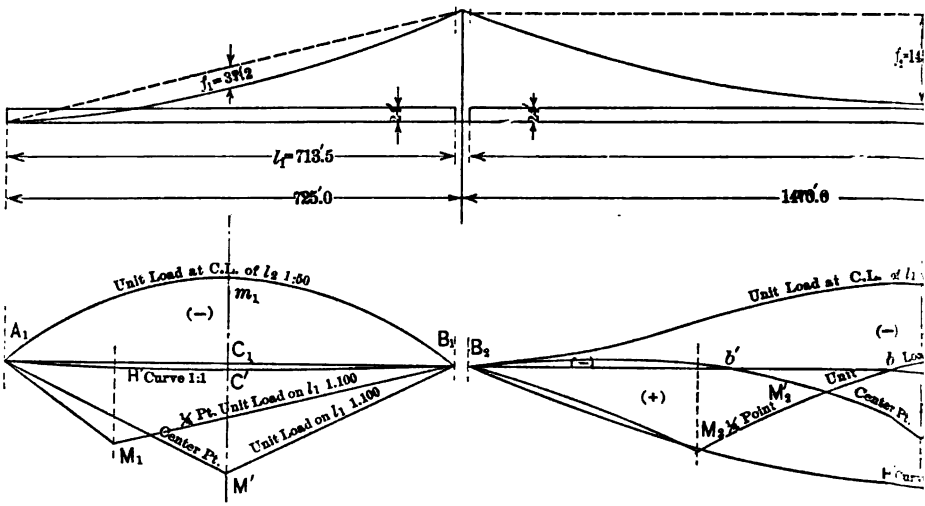
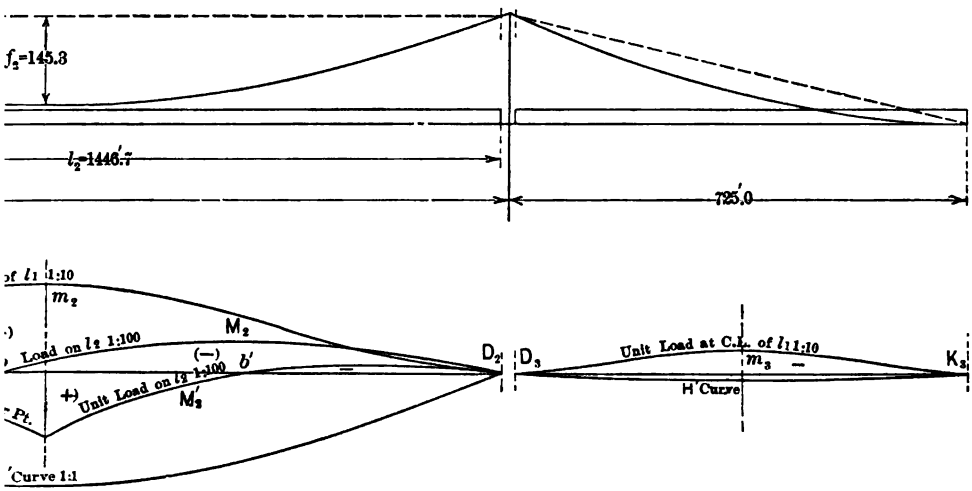


PLATE I.—Influence L



ce Lines for Moments.

span in which the greatest negative moment is sought as indicated by the influence lines for the points considered.

The greatest positive moments in any one span will be produced by the amounts and positions of moving load indicated by the moment influence lines for the points considered with no loading on the other two spans, as loads on those two spans would produce negative moments at those points.

If it is desired to determine the values of other moments for other positions of loading than those giving the greatest values, the procedures set forth above in detail show clearly what combinations must be made. No other considerations need be employed for this purpose than those which have been illustrated.

Shears.

The same data with the same lengths of spans will be used as in the preceding application of moment influence lines.

The ordinary method employed will first be illustrated, as was done in the case of moments, after which the complete influence lines will be constructed.

If a unit load be assumed in Eq. 212, that equation will take the form:

$$S = -\left(\frac{q}{\frac{4f_1}{l_1}\left(1 - \frac{2x}{l_1}\right)} + H'\right)\frac{4f_1}{l_1}\left(1 - \frac{2x}{l_1}\right) \quad . \quad 226.$$

For the other segment of the span, when z is greater than x, Eq. 213 may take the form:

$$S = \left(\frac{1-q}{\frac{4f_1}{l_1}\left(1 - \frac{2x}{l_1}\right)} - H'\right)\frac{4f_1}{l_1}\left(1 - \frac{2x}{l_1}\right) \quad . \quad 227.$$

In this case $\frac{4f_1}{l_1} = .2088$, and H' is to be taken from the tabulation following Eq. 224.

It is at once evident that the first terms in the first parentheses of the second members of Eqs. 226 and 227 are ordinates

of a straight line, as x is the only variable in either term. Hence, if x be made equal to $\frac{l_1}{4}$ in each of those terms, there will result for Eq. 226:

$$\frac{q}{\frac{4f_1}{l_1}\left(1 - \frac{2x}{l_1}\right)} = \frac{q}{\frac{2f_1}{l_1}} \quad . \quad . \quad . \quad . \quad . \quad 228.$$

and for Eq. 227:

$$\frac{1-q}{\frac{4f_1}{l_1}\left(1 - \frac{2x}{l_1}\right)} = \frac{1-q}{\frac{2f_1}{l_1}} \quad . \quad . \quad . \quad . \quad 229.$$

If K, Plate II, in the span l_1 of Plate B be at the quarter point measured from A, and if the unit load be placed at the same point, *i.e.*, if $q = .25$, then will $P\,K_1 = 2.394$ be laid off equal to the first of the above expressions, while $K_1\,Q_1 = 7.184$ will be laid off in the opposite direction to the same scale, representing Eq. 229 for the same value of q. The straight line $A\,P$ represents graphically the first term of the first parenthesis in Eq. 226, while $Q_1\,B_1$ represents the first term of the first parenthesis in the second member of Eq. 227. Furthermore as the ordinates of the curve $A_1\,T_1$ represent the values of H' taken from the tabulation following Eq. 241, it is clear that $P\,T_1$ represents the first parenthesis in the second member of Eq. 226, since it is the sum of H' and the term computed above, while $T_1\,Q_1$ will represent the first parenthesis in the second member of Eq. 227 as it is the difference between $K_1\,Q_1$ and H'. The shear $P\,T_1$ is negative and belongs to a point immediately adjacent to K_1 but to the left of it, while $T_1\,Q_1$ is the positive shear immediately adjacent to K_1 but to the right of it. Any ordinate drawn from the curve $A_1\,T_1\,B_1$ to either of the lines $A_1\,P$ or $Q_1\,B_1$ if multiplied by $\frac{4f_1}{l_1}\left(1 - \frac{2x}{l_1}\right)$ will represent the shear at the point corresponding to the value of q which locates it. The lines thus constructed will give in the manner indicated the shear at the quarter point for any position of the unit load in the entire span.

Corresponding lines may be constructed for any other point

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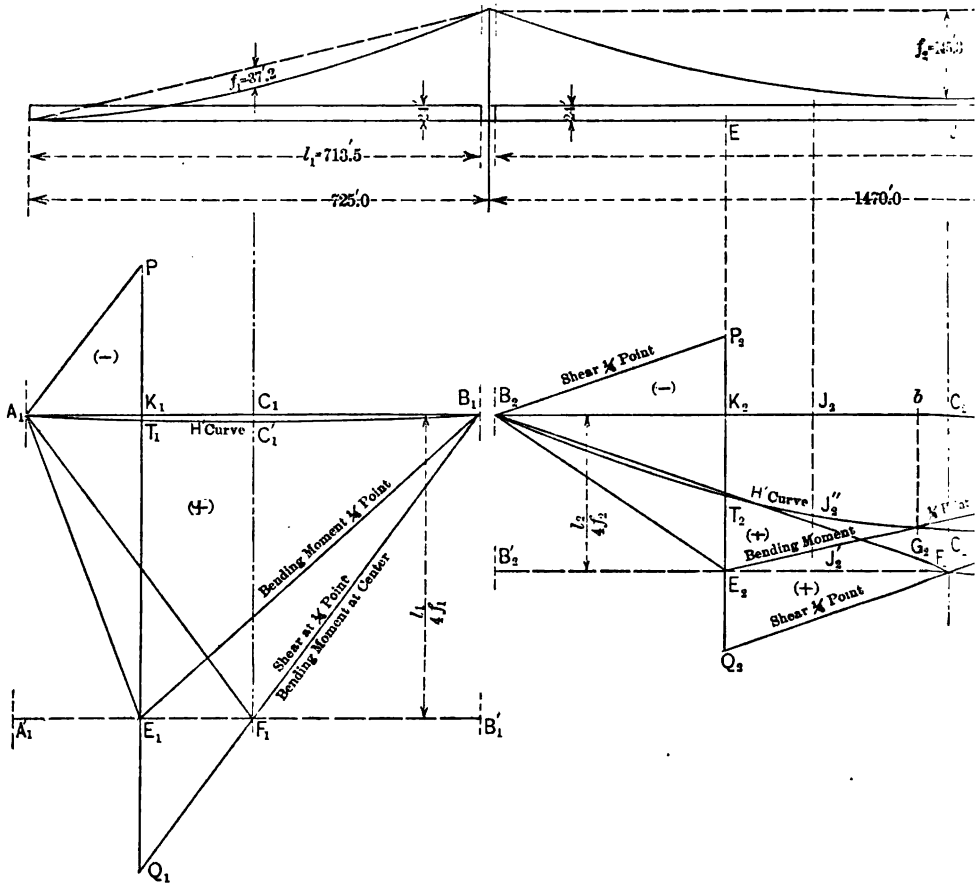
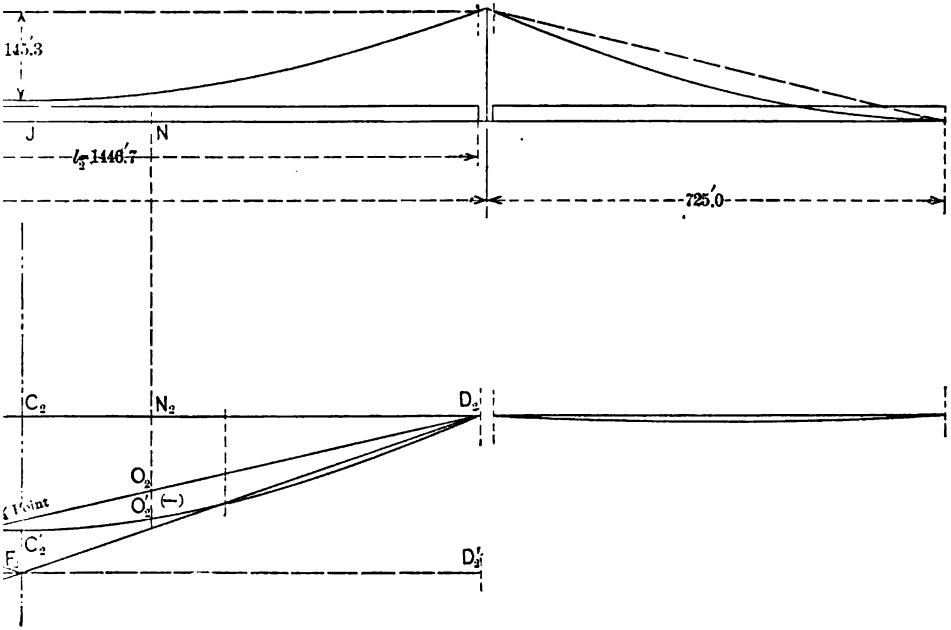


PLATE II.—Influence Li



e Lines for Moments.

or section in the span in precisely the same manner, thus giving influence lines for all points. After making the constructions indicated it is necessary to multiply the scaled distance by the quantity shown outside of the parentheses in the second members of Eqs. 226 and 227.

It has already been shown in the second paragraph following Eq. 214 that the shears produced at the centre of the span by loads at all points in the span are precisely the same as in the case of a non-continuous beam simply supported at each end.

A graphic construction similar to that made above for the side span l_1 will be shown for the quarter point K_2 in the span l_2, Plate II. Eqs. 226 and 227 are directly applicable after changing the subscripts 1 to 2. By making $x = \frac{l_2}{4}$ and $q = .25$, the first term in the first parenthesis of the second member of Eq. 226 becomes 1.25 and $P_2 K_2$ is laid off with that value vertically from K_2. With the same values of x and q the first term of the first parenthesis in the second member of Eq. 227 becomes 3.75, and $K_2 Q_2$ is laid off downward from K_2 with that value. Straight lines are then drawn from P_2 and Q_2 to B_2 and D_2, respectively. The curved line $B_2 T_2 D_2$ represents H' as given for all points by the tabulation following Eq. 215. The shear at the quarter point K_2 caused by the unit load at any point of the span will therefore be given by taking the difference between, or the sum of, the ordinates, at the point considered, drawn to the straight line $Q_2 D_2$ and the curve $B_2 T_2 D_2$ and multiplying the resulting quantity by $\frac{4 f_2}{l_2}\left(1 - \frac{2x}{l_2}\right)$. Similar influence lines may be constructed for any other point in the span l_2, but the operation has been sufficiently illustrated by the constructions made for the quarter point.

Although these constructions will enable the positions of loading to be determined for the greatest shears at any points in the span, together with the actual values of those shears, it is believed to be simpler and clearer to deal with influence lines showing the actual shears themselves, especially as no greater

labor is involved and perhaps less. Such constructions will now be made.

It will be convenient to use the simpler general forms of the shear S, Eqs. 212 and 213, the former when z is less than x and the latter when z is greater than x. As the construction will first be made for the side span l_1, the subscript 2 will be changed to 1 in Eqs. 212 and 213. The quarter point K_1 of the span l_1, Plate III, is to be considered; hence, $x = \frac{l_1}{4}$. By placing this value in the two equations named above there will result for z less than x, and for a unit load:

$$S = -q - \frac{2f_1}{l_1} H' \quad . \quad . \quad . \quad . \quad . \quad 230.$$

when z is greater than x:

$$S = (1 - q) - \frac{2f_1}{l_1} H' \quad . \quad . \quad . \quad . \quad 231.$$

The data already given show that $\frac{2f_1}{l_1} = .1044$. If q be given the values indicated in the table below, Eqs. 230 and 231 will give the values of the shears S also shown in the table.

SHEAR S

q	Span l_1	q	Span l_1
0	0	.5	+ .489,968
.1	− .103,15	.6	+ .390,446
.2	− .205,958	.7	+ .291,842
.25	− .257,148	.8	+ .192,852
	+ .742,852	.9	+ .094,042
.3	+ .691,842	1.0	0
.4	+ .590,446		

These tabulated values are laid off on Plate III at a scale showing them five times as large as given in the table, the negative shears being laid off upwards from the line $A_1 B_1$, and the positive shears downwards, $K_1 S_1$ being $5 \times .257 = 1.285$,

and $K_1 S'_1$ representing $5 \times .743 = 3.715$. The line representing the shears at the quarter point of the span for all positions of the unit load in that span will then be $A_1 S_1 K_1 S'_1 B_1$. All loads on the segment $A_1 K_1$ of the span will therefore produce negative or upward shears at the quarter point K_1, while all loads between K_1 and B_1 will produce positive or downward shears. This influence line represents at a glance, with the least effort, complete shear conditions produced at the quarter point of the span by any or all loads. The lines $A_1 S_1$ and $S'_1 B_1$ are nearly straight.

By making $P = o$ in either Eq. 212 or 213 there will result for the unloaded span l_2:

$$S = - .1044 H' \quad . \quad . \quad . \quad . \quad . \quad 232.$$

Also for the unloaded span l_2 for which $\frac{f_2}{l_2} = .1$:

$$S = - .4 H' \quad . \quad . \quad . \quad . \quad . \quad . \quad 233.$$

In these two equations H' will clearly have values due to the position of the unit load in the span l_1 as given in the tabulation following Eq. 224.

A similar construction is readily made for the span l_2, also as shown on Plate III. Again using the simple general equations 212 and 213, remembering in this case that $\frac{f_2}{l_2} = .1$, and that for a unit load $P = 1$, there will result for the quarter point K_2 of the span:

For z less than x:

$$S = - q - .2 H' \quad . \quad . \quad . \quad . \quad . \quad 234.$$

For z greater than x:

$$S = 1 - q - .2 H' \quad . \quad . \quad . \quad . \quad . \quad 235.$$

Again, giving q the values shown in the tabular statement

below, the corresponding values of shear, also shown in the table, will result:

SHEAR S

q	Span l	q	Span l
0	0	.5	+ .141,8
.1	− .212,44	.6	+ .058,9
.2	− .412,74	.7	+ .008,76
.25	− .505,22	.8	− .012,74
	+ .494,78	.9	− .012,44
.3	+ .408,76	1.0	0
.4	+ .258,9		

Laying these shears off at right angles to the horizontal line $B_2 C_2$, Plate III, upward when negative and downward when positive, the influence line $B_2 S_2 K_2 S'_2 D_2$ will be found. This influence line shows at a glance all the shear conditions in the span l_2 affecting the quarter point K_2. All loads on the two segments $B_2 K_2$ and $O D_2$ will produce negative or upward shears while all those on the segment $K_2 O$ will produce positive or downward shears at the quarter point K_2. If $P = 0$ in either Eq. 212 or 213 the value of the shear S in the unloaded spans l_1 and l_3 will take the value:

$$S = - .2 H' \quad . \quad . \quad . \quad . \quad . \quad 236.$$

In this equation H' will have the value shown in the tabulation following Eq. 215.

The end shears R_1 and R_2 in the spans l_1 and l_2 for a unit load at any point in either one of those spans are found by making $x = 0$ in Eq. 213, because in such a case z is always greater than x. By making $x = 0$ and $P = 1$ in that equation there will result after substituting $\frac{f_1}{l_1} = .0522$ for $\frac{f_2}{l_2}$:

$$R_1 = 1 - q - .2088 H' \quad . \quad . \quad . \quad . \quad 237.$$

Making $\frac{f_2}{l_2} = .1$, $x = 0$, and $P = 1$ in Eq. 213, there will result

for the span l_2:

$$R_2 = 1 - q - .4\,H' \quad . \quad . \quad . \quad . \quad 238.$$

Taking the value H^1 for R_1 from the tabulation following Eq. 224, and the value of H^1 for R_2 from the tabulation following Eq. 215, the following tabulated values of R_1 and R_2 will result:

q	R_1	R_2
0	+ 1	+ 1
.1		+ .675,12
.2	+ .788,084	+ .374,52
.4	+ .580,893	+ .082,2
.5	+ .479,936	− .216,4
.6	+ .380,893	− .282,2
.8	+ .188,084	− .225,48
.9		− .124,88
1.0	0	0

These numerical quantities are laid off vertically three times their full size, positive downward and negative upward, from the horizontal line $A_1 B_1$ in the span l_1 and $B_2 D_2$ in the span l_2, Plate III, giving the curved line $R_1 B_1$ in the former span and $R_2 O' D_2$ in the latter. All reactions are positive in the span l_1, but part are positive and part negative in span l_2. In the latter span all loads between B_2 and O' cause positive or upward reactions at B_2, while all loads between O' and D_2 cause negative or downward reactions at the same extremity B_2 of the span. The maximum anchorage, *i.e.*, negative reaction, required in the span l_2 by any single load can be scaled as the greatest ordinate between the negative part of the reaction curve and $O' D_2$. The greatest possible negative reaction at B_2, *i.e.*, the greatest amount of anchorage, will be found by loading only the entire segment $O' D_2$ of the span, with the other two spans fully loaded. Similarly the greatest possible positive reaction for upward support required at B_2 will be found by loading only the entire segment $B_2 O'$, the other two spans being wholly unloaded.

These constructions for the shears, including the end shears

or reactions, illustrate completely the influence line procedure needed to determine the position and amounts of loading causing the greatest possible shears at any section. It must be remembered, only, that loading on one span will produce negative or upward reactions on the two unloaded spans.

It should be observed in passing that, as usual with influence lines, the areas between such lines and the horizontal span line represent total shears at the points for which they are drawn. It is only necessary to multiply such areas expressed in full size dimensions, *i.e.*, square feet, by the uniform load per running foot in order to obtain the total shears for the case of a uniform load per linear foot.

The application in this article of the preceding analysis to an actual stiffened suspension bridge structure relates to a stiffening-truss of constant depth and with a uniform moment of inertia of cross-section of truss for each span, the moment of inertia for the side span being different from that belonging to the trusses of the main span. It will seldom be necessary to follow any different procedure for the moments of inertia of stiffening-truss sections, although the analysis indicates clearly and fully the proper modifications to be made in the formulæ should such modifications become necessary. Such modifications would involve complications and additional labor in computations which would seldom be justified, except for varying depths of trusses.

Although the item in the quantity D of Eq. 232 of this article depending upon the work expended in stretching the suspenders is not separately computed in the preceding numerical work, it does not exceed about 3% of the total value of D, and it could therefore be omitted without material error. Its inclusion in Eq. 70 of Art. 46, however, adds no sensible amount of labor to the computations, and as its use contributes somewhat to accuracy it has been inserted in the analysis and employed in the computations.

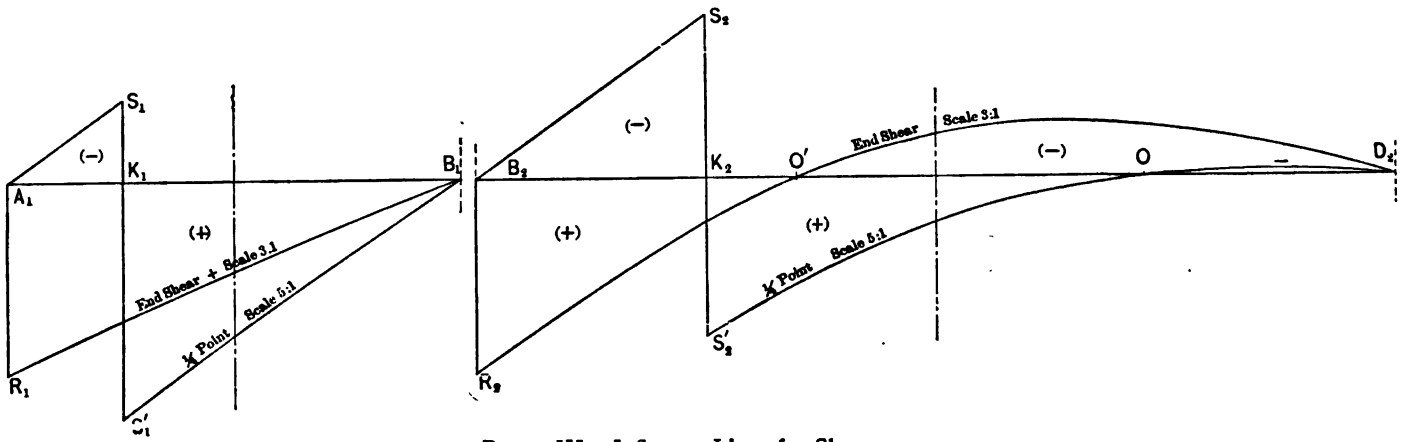


PLATE III.—Influence Lines for Shears.

Art. 60.—Numerical Computations.

In the preceding article it has been seen that the numerical computations for the horizontal component H' of the cable stress will be much aided by a table of values of the following expression: $q\,(1 - q^2\,(2 - q)) = Q$. As this expression has the same value for q and $1 - q$ it is necessary to compute results from $q = 0$ to $q = .5$ only. By varying q, therefore, from 0 to .5, the following tabulation will result:

q	Q	q	Q
.0 or 1.	0		
.01 " .99	.009,998	.26 or .74	.229,418
.02 " .98	.019,984	.27 " .73	.235,948
.03 " .97	.029,947	.28 " .72	.242,243
.04 " .96	.039,875	.29 " .71	.248,295
.05 " .95	.049,756	.30 " .70	.254,1
.06 " .94	.059,581	.31 " .69	.259,653
.07 " .93	.069,338	.32 " .68	.264,95
.08 " .92	.079,017	.33 " .67	.269,985
.09 " .91	.088,608	.34 " .66	.274,755
.10 " .90	.098,100	.35 " .65	.279,256
.11 " .89	.107,484	.36 " .64	.283,484
.12 " .88	.116,751	.37 " .63	.287,436
.13 " .87	.125,892	.38 " .62	.291,107
.14 " .86	.134,905	.39 " .61	.294,496
.15 " .85	.143,756	.40 " .60	.297,6
.16 " .84	.152,463	.41 " .59	.300,416
.17 " .83	.161,009	.42 " .58	.302,941
.18 " .82	.169,386	.43 " .57	.305,174
.19 " .81	.177,585	.44 " .56	.307,113
.20 " .80	.185,600	.45 " .55	.308,756
.21 " .79	.193,423	.46 " .54	.310,103
.22 " .78	.201,047	.47 " .53	.311,151
.23 " .77	.208,464	.48 " .52	.311,9
.24 " .76	.215,670	.49 " .51	.312,35
.25 " .75	.222,656	.50	.312,5

Art. 61.—Greatest Bending Moment in Non-Continuous Stiffening-Truss Due to Uniform Load from One End of Span.

The formulæ and the proper procedure in this case are fully set forth in Art. 52. Eq. 125, giving the horizontal component

H' of the moving load cable stress, must be used in Eq. 129, which gives the location at which the greatest bending moment in the stiffening-truss exists for a given length x_1 of uniform moving load extending from the same end of the span from which x is measured. Eq. 130 of Art. 52 gives the greatest bending moment located by the value of x given by Eq. 129.

The stiffening-truss for the centre or main span of the Manhattan Bridge taken in the application of formulæ already set forth in Art. 60 will be employed here, consequently the same data will be used as in Art. 60. In this case, however, no end spans will be supposed to exist, and the backstay cables will therefore be assumed to be straight. Each of the backstay spans will be taken as 440 feet in length from the centre of the tower to the point where the backstay cable enters the anchorage, thus making the angle of inclination of the backstay cable to a horizontal at the top of the tower the same as for the main cable at the same point. The tangent of that inclination will be .4 and the secant 1.0773. The following data may be conveniently reproduced from Art. 60, $F_0 = 275$ square inches; $F_2 = .1$ square inch; $F_3 = 400$ square inches.

$$I = 44{,}000 \text{ (square inches} \times \overline{\text{ft.}}^2\text{)}.$$

$$l = 1{,}447 \text{ feet.}$$

$$f = 145 \text{ feet.}$$

Introducing the preceding data and other data of Art. 60 in the denominator of Eq. 74 of Art. 46, that denominator, designated by D', will have the following value:

$$D' = 111{,}158 \quad . \quad . \quad . \quad . \quad . \quad 239.$$

Remembering that, as these computations are for a uniform moving load only, both k and w_0 must be made zero in Eq. 74 in establishing H' to be substituted in Eq. 129.

Successive uniform loads covering one-tenth, two-tenths, three-tenths, etc., of the span will be employed, and the value of x, Eq. 129 of Art. 52, must be found for each such progressive load. Instead of taking tenths of the actual span it will be more

convenient to take values of $\frac{x_1}{l_2}$ as .1, .2, .3, etc.

Substituting the values of the varying ratio $\frac{x_1}{l}$ in Eq. 129, in the manner indicated and determining both the corresponding values of x and the greatest bending moment M_m, Eq. 130, Art. 52, the following tabulated numerical values will result:

$\frac{x}{l}$	x	M_m
.1	.0086 l_2	− .000,359,9 $p_2 l_2^2$
.2	.149 "	− .010,136,5 " "
.3	.198 "	− .015,889,5 " "
.4	.237 "	− .019,202,9 " "
.5	.270 "	− .019,834,2 " "
.6	.302 "	− .018,391,8 " "
.7	.337 "	− .015,738 " "
.8	.386 "	− .013,075,8 " "
.9	.454 "	− .011,225,2 " "
1.0	.5 "	− .010,96 " "

It is simpler to express x as fractions of l_2 as it is also simpler to express the bending moments M as fractions of $p_2 l_2^2$, and the

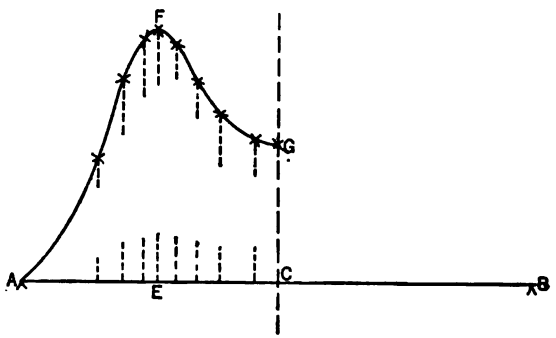


FIG. 17.

numerical results are so expressed. The numerical quantities are actual values for a unit length of span with a unit moving load per linear foot.

Fig. 17 shows graphically the results of these computations

The numerical quantities in the column x of the table are laid off as horizontal ordinates in Fig. 17 from A towards C and at the point so determined the corresponding numerical quantity in the column M_m is laid off vertically at one hundred times the value shown in the table. The distance $A\,B$ represents the span l_2. Hence, as an illustration, $A\,E$ for which $\frac{x_1}{l_2} = .5$ was taken 27 hundredths of the span, or $A\,E = .27\,l_2$. $E\,F$ was then laid off vertically to represent M_m as 1.98 inches, which is one hundred times the value of the corresponding decimal in the table. Other points of the greatest moment curve $A\,F\,G$ were similarly constructed, $G\,C$ representing the greatest moment for the load over the whole span, for which $\frac{x_1}{l_2} = 1$, or $x = .5\,l_2$ and $M_m = .01096$ as a unit quantity, *i.e.*, $G\,C = 1.096$. The curve shows that the greatest bending moment in the entire span for all positions of the continuous moving load is found by loading one-half the span with a continuous load, the greatest bending moment itself being equal to $.019{,}834\ p_2\,l_2^2$. If a curve were constructed on a much larger scale it might appear that the greatest possible moment would be a little larger than that indicated by the above table and that a little more or less than half the span should be loaded, but the difference would be so small that it may be neglected for the present purpose.

Obviously the curve $A\,F\,B$ may also be constructed for the half $C\,B$ of the span, each half of the curve being symmetrical with the vertical centre line $G\,C$.

It is clear that the numerical computations in this case are much expedited by the table of numerical values found in Art. 52 between Eqs. 130 and 131. In fact, these tabulated values make the labor of computation immaterial and enable the results to be obtained with ease and rapidity.

It is interesting to observe that Eq. 52 of Art. 11 shows that the greatest possible bending moment in the stiffening-truss, by the old or static method of computation, is:

$$M_m = .018{,}52\ p_2\,l_2^2.$$

Again, Eq. 16 of Art. 16 shows that if the non-continuous stiffening-truss has a hinge at the centre of the span, making the problem statically determinate, the greatest possible bending moment in the span will be:

$$M_m = .018{,}837\ p_2 l_2^2.$$

Finally, the above more nearly exact method in which the results obtained depend upon the elastic properties of the materials gives:

$$M_m = .019{,}834\ p_2 l_2^2.$$

The last result is from 6 to 7% larger than the other two, which is not a great difference. It may perhaps be questioned whether the labor and complication of the more refined demonstration and computations are fully justified by the relatively small difference in the greatest bending moments found. The most important difference, however, is not that corresponding to the greatest bending moments in the span, but to the values of the centre bending moments for load covering the entire span, or nearly the entire span. If the uniform moving load covers the entire span there will be no centre bending moment by the old or static theory, nor at any other point of the span, but the more nearly exact method shows that when the entire span is covered, the bending moment at the centre of the span is distinctly more than half the greatest bending moment.

Art. 62.—Special Case of End Spans.

The preceding analysis is perfectly general and it covers the case of Fig. 18, in which the spans l_1 and l_3 are the extensions of the stiffening-trusses into the backstay spans where there are no suspenders. A and B are the anchorages for the cable, while G and K would ordinarily be steel bents attached at their tops to the ends of the stiffening-trusses and at their bottoms to suitable masonry piers, all so designed as to resist either positive or negative reactions.

All integrals requiring the moving load bending moment M_b may at once be found for all spans by Eqs. 52 to 57 of Art. 44.

If there is no moving load on any one span $\Sigma P z\ (1 - q)$ for that span disappears.

The design, construction, and erection of the three spans of the stiffening-trusses should be such that the entire dead load of the main span may be carried by the cable at mean temperature. This requires that each of the three spans of trusses must be

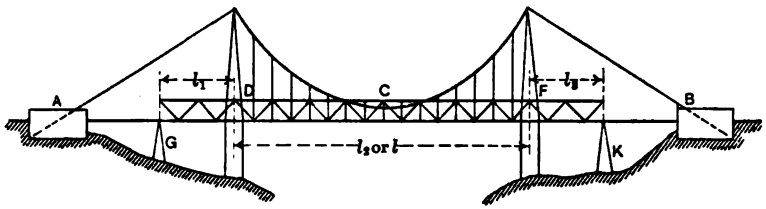


FIG. 18.

erected in place complete before being connected together over the main piers at D and F. The trusses in the spans l_1 and l_3 should be erected as simply supported non-continuous spans, swinging free from their falseworks and carrying all their dead load before the connections with the centre span l_2 are made. If the side and centre spans are connected under other conditions much uncertainty as to the amount of dead load carried by the suspenders will exist even at mean temperature.

Dead Load Stresses in Spans l_1 and l_3.

Under the conditions prescribed above, the dead load stresses in the spans l_1 and l_3 will be those only in simply supported, non-continuous spans. They should be computed and tabulated to be combined with moving load stresses.

Stresses Due to Moving Load in Spans l_1 and l_3.

Moving load will pass on the spans l_1 and l_3 after the three spans are made continuous, and that moving load will produce stresses in the trusses of the span l_2 which, causing deflection of that span, will induce change in the uniform suspender pull. If that change in suspender pull is s_2 per foot of span, the action of moving loads in either l_1 or l_3 is represented by the application

of the loads P in those spans and of the uniform downward load s_2 over the entire span l_2. Remembering that $\overset{1}{\Sigma} P\,(1-q^2)\,q$ will now become $\frac{s_2 l_2}{4}$, the moment M_3 at F, Fig. 18, will be given by Eq. 14, Art. 42:

$$M_3 = l_2 \frac{a^2 \overset{1}{\Sigma} P(1-n^2)n - (2a+r)\frac{s_2 l_2}{4} - 2b^2(a+r)r^1 \overset{1}{\Sigma} P(1-m^2)m}{4\,(a+r)\,(b\,r^1+1) - r} \quad . \quad 259.$$

Then, from Eq. 15, Art. 42:

$$M_2 = -\,l_2 \frac{\frac{r\,M_3}{l_2} + a^2 \overset{1}{\Sigma} P\,(1-n^2)\,n + r\,\frac{s_2 l_2}{4}}{2\,(a+r)} \quad . \quad 260.$$

The four moving load reactions at G, D, F, and K will then become:

$$R_1 = \overset{1}{\Sigma} P\,(1-n) + \frac{M_2}{l_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 261.$$

$$R_2 = \overset{1}{\Sigma} P\,n + \frac{s_2 l_2}{2} + \frac{M_3 - M_2}{l_2} - \frac{M_2}{l_1} \quad . \quad . \quad . \quad 262.$$

$$R_3 = \overset{2}{\Sigma} P\,(1-m) + \frac{s_2 l_2}{2} + \frac{M_2 - M_3}{l_2} - \frac{M_3}{l_3} \quad . \quad 263.$$

$$R_4 = \overset{2}{\Sigma} P\,m + \frac{M_3}{l_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 264.$$

The reactions R_1 and R_4 are the end shears at G and K, Fig. 18. The shear at the other end of l_1 (at D) is:

$$S_D = {}^{1}P - R_1 = \overset{1}{\Sigma} P\,n - \frac{M_2}{l_1} \quad . \quad . \quad . \quad . \quad 265.$$

And adjacent to F on the right hand of that point:

$$S_F = {}^{2}P - R_4 = {}^{2}P\,(1-m) - \frac{M_3}{l_3} \quad . \quad . \quad . \quad 266.$$

Change s_2 in Suspender Pull.

The preceding general values of moments, reactions, and shears for this case are necessary for finding the final truss stresses in the three spans, but it is next essential to determine the uniform

change (s_2 per foot of span) of the suspender pull in span l_2 due to loads on l_1 and l_3. This latter condition is equivalent to a uniform downward vertical load s_2 over the entire span which will cause bending (and stresses) in all three spans, even if no loads should act in spans l_1 and l_3, as with temperature stresses. When moving load passes on either l_1 or l_3, or on both, there will be upward deflection of the trusses in span l_2. This will relieve the cable of load s_2 per linear foot and give to the stiffening-truss as an ordinary continuous truss an equal amount of uniform load, thus making the upward deflection of the cable added to the contraction of the suspenders (due to release of load) equal to the upward deflection of the stiffening-truss.

If the moving load on the spans l_1 and l_3 is symmetrical in amount and location in reference to the span l_2 it is clear that the upward deflection of the truss at the centre of span l_2 (and of the cable and contraction of the suspenders) will be in proportion to the amount of that symmetrical loading. Hence, if the amount of symmetrical loading required to relieve the cable and suspenders of all own weight of stiffening-trusses and floor, *i.e.*, w_s per linear foot of span l_2, be determined, the loading on spans l_1 and l_3 necessary to relieve the cable to the extent of s_2 per linear foot of span will be proportionate to that relief.

To shorten and simplify the analysis the usual case of $l_1 = l_3$ will be assumed, and l will be written for l_2.

For this particular purpose it will further be assumed that any moving load on either l_1 or l_3 is uniformly distributed over their entire lengths. That assumption will simplify computations and lead to no material error in the results actually to be used, as the latter belong chiefly to the condition of moving load covering entirely either l_1 or both l_1 and $l_3 = l_1$.

Let the combined shortening of the suspender at the centre of span l and the upward deflection of the cable at the same point due to relief from all stiffening-truss load w_s per lin. ft. be determined by the methods of Arts. 35 and 36, taking into account the shortening of the backstay cables; and let it be designated by u''_c.

The formulæ of Arts. 35 and 36 give at once those parts of

u''_c due to the shortening of the cables and backstays and to the movements of the saddles at the tops of the towers. If t'' is the intensity of stress in the suspenders due to any suspender load s_2 (w_s being the greatest value), and if v is the length of shortest suspender at the vertex of the cable, then that part of u''_c due to the shortening of the suspender v is:

$$\frac{t''}{E} v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 267.$$

The upward deflection u''_c thus becomes completely known, as will be shown by the following analysis.

If the uniform moving load p_1 rests upon each of the end spans while the uniform load s_2 rests upon the centre span, l, Eqs. 259 and 260 show that the bending moment at each end of that centre span will be:

$$M_3 = M_2 = -l \frac{a^2 p_1 l_1 + r s_2 l}{4(2a+3r)} \quad . \quad . \quad . \quad . \quad 268.$$

The reactions at the two abutments and at the two piers will then take the following forms by the aid of Eqs. 261, 262, 263, and 264:

$$R_1 = \frac{p_1 l_1}{2} + \frac{M_2}{l_1} \quad . \quad . \quad . \quad . \quad . \quad 269.$$

$$R_2 = R_3 = \frac{p_1 l_1}{2} + \frac{s_2 l}{2} - \frac{M_2}{l_1} \quad . \quad . \quad . \quad 270.$$

$$R_4 = R_1 = \frac{p_1 l_1}{2} + \frac{M_3}{l_1} \quad . \quad . \quad . \quad . \quad 271.$$

It will now be necessary to determine the deflection (upward) of the centre of span l under the loads used in connection with Eq. 268.

Taking moments about any point distant x from the left end of span l:

$$E\ I \frac{d^2u}{dx^2} = M_2 + \frac{s_2 l}{2} x - \frac{s_2 x^2}{2} \quad . \quad . \quad . \quad 272.$$

Integrating between the limits x and $\frac{l}{2}$ and remembering that for $x = \frac{l}{2}$, $\frac{du}{dx} = 0$:

$$EI\frac{du}{dx} = \frac{M_2}{2}(2x - l) + \frac{s_2}{4}\left(lx^2 - \frac{2x^3}{3} - \frac{l^3}{6}\right) \quad . \quad 273.$$

Again integrating between the limits x and 0, remembering that $u = 0$ for $x = 0$:

$$EIu = \frac{M_2}{2}(x^2 - xl) + \frac{s_2}{4}\left(\frac{lx^3}{3} - \frac{x^4}{6} - \frac{l^3x}{6}\right) \quad . \quad 274.$$

When the uniform load s_2 is equal to the own weight of the stiffening-truss and tracks or flooring carried by it the deflection at the centre of the span where $x = 2l$ becomes u''_c, Eq. 274 then gives:

$$u''_c = -\frac{M_2 l^2}{8EI} - \frac{5 w_s l^4}{384 EI} \quad . \quad . \quad . \quad . \quad 275.$$

It is necessary to determine the value of the uniform load p_1 resting on the span l_1 when the relief of suspender pull s_2 becomes equal to w_s. Hence placing the value of M_2 from Eq. 268 in Eq. 275, the following value of p_1 will result:

$$p_1 = \frac{w_s l}{12 a^2 l_1}(10a - 3r) + \frac{32(2a + 3r)EIu''_c}{a^2 l_1 l^3} \quad . \quad 276.$$

It has already been indicated that the centre deflection u''_c must be equal to the upward deflection of the vertex of the cable due to the shortening of the latter, under the relief of the uniform load w_s, added to $\frac{t''}{E}v$.

The shortening of the cable resulting from relieving it of the own weight or dead load w_s per linear foot must obviously be precisely the same as the lengthening which would take place under the imposition of this same load w_s. Furthermore, a corresponding shortening or lengthening, as the case may be, of the backstays will exist concurrently with the shortening or

lengthening of the main cable. The upward deflection of the vertex of the cable resulting from the shortening of the main cable and the two backstays must next be found.

It is necessary to recognize both the cable with constant cross-section F (like a wire cable) and with a variable cross-section $F_0 \frac{dL}{dx}$ in finding the vertical deflection of the vertex of the cable due to the relief of the suspender pull.

If H_1 is the horizontal component of cable stress due to the uniform load w_s on the entire centre span l:

$$H_1 = \frac{w_s l^2}{8f}, \text{ and the cable stress at any point } = \frac{w_s l^2}{8f}\frac{dL}{dx}.$$

Hence the total stretch of the wire cable (constant cross-section) is:

$$\Delta L = 2\int_0^{\frac{l}{2}} \frac{H_1}{EF}\frac{dL^2}{dx^2}\,dx = \frac{w_s l^2}{8fEF}\,2\int_0^{\frac{l}{2}} \frac{dL^2}{dx^2}\,dx$$

$$= \frac{w_s l^3}{8fEF}\left(1 + \frac{16}{3}\frac{f^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad 277.$$

The stretch of the cable with variable section proportioned to the total cable stress will be:

$$\Delta L = 2\int_0^{\frac{l}{2}} \frac{H_1}{EF_0}\frac{dL}{dx}\,dx = \frac{w_s l^2}{8fEF_0}\,2\int_0^{\frac{l}{2}} \frac{dL}{dx}\,dx$$

$$= \frac{w_s l^3}{16fEF_0}\left[\sqrt{1 + \frac{16f^2}{l^2}} + \frac{l}{4f} log\left(\frac{4f}{l} + \sqrt{1 + \frac{16f^2}{l^2}}\right)\right] \quad . \quad 278.$$

If $\frac{f}{l} = n$, Eq. 82 of Art. 35 shows that the change in f corresponding to the change ΔL of cable length is:

$$\Delta f = \frac{15}{16\,(5n - 24n^3)}\,\Delta L \quad . \quad . \quad 279.$$

The total stress in a backstay is:

$$H_1 \sec \alpha_1 = \frac{w_s l^2}{8f} \sec \alpha_1.$$

Hence the change in length of backstay due to that total stress is:

$$\Delta L_s = \frac{w_s l^2 l_1}{8 f F_1 E} \sec^2 \alpha_1.$$

The horizontal component of ΔL_s is:

$$\Delta L_s \cos \alpha_1 = \frac{w_s l^2 l_1}{8 f F_1 E} \sec \alpha_1 = \Delta l_1.$$

This quantity is the movement of the saddle on the top of a tower away from the centre span when the suspender pull in the latter is relieved. The change in length of the centre span is obviously:

$$\Delta l = 2 \Delta l_1.$$

Placing $f \div l = n$, and indicating by $-\Delta f'$ the upward deflection due to the increase of span Δl, Eq. 89 of Art. 35 shows:

$$\Delta f' = \frac{15 - 8 (5 n^2 - 36 n^4)}{16 (5 n - 24 n^3)} \Delta l \quad . \quad . \quad . \quad 280.$$

The complete expression for the centre upward deflection u''_c now becomes:

$$u''_c = \Delta f + \Delta f' + \frac{t''}{E} v \quad . \quad . \quad . \quad . \quad 281.$$

If the numerical value of u''_c determined by Eq. 281 be placed in the second term of the second member of Eq. 276, the latter equation will give at once the value of the uniform load p_1, which must rest on the whole of the spans l_1 and $l_3 = l_1$ in order to give an upward deflection at the centre of l just sufficient to relieve the cable of the weight w_s of the stiffening-truss and tracks or roadway platform carried by it.

If the uniform load p_1 rests upon the whole of either of the spans l_1 or l_3 while no moving load rests upon the other of those two spans, the upward centre deflection of the span l will be one-half of u''_c, since the moving load on the whole of one of those spans will produce just as much deflection at the centre of span l as that on the other. Similarly, any amount of moving load on either one of those spans will produce just as much upward

centre deflection in l as the same load similarly placed on the other.

Obviously if any other uniform moving load p' covers the whole of l_1 and $l_3 = l_1$, the upward deflection produced by such loading at the centre of l and the relief of suspender pull in that span will be, respectively:

$$u'_c = \frac{p'}{w_s} u''_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 282.$$

and:

$$s'_2 = \frac{p'}{p_1} w_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 283.$$

Since the same uniform load over the whole of either span l_1 or l_3 will cause half the upward centre deflection in span l that the same uniform load on both those spans will produce, it is clear that a uniform load over either span l_1 or l_3 will produce half the relief of suspender pull in span l that would arise from the same uniform load on both of those spans.

These formulæ will enable all bending moments and stresses in span l caused by moving load on the side spans l_1 and l_3 to be found. The greatest possible negative bending moments in span l will occur when the moving load covers the whole of both the side spans.

Moving Load in Main Span $l_2 = l$.

In treating the moving load moments and stresses for this special case it is now necessary to consider the side spans with no moving load whatever in the determination of the integral $M_b \, dx$ while the main span l of the stiffening-truss may have any such load.

Eqs. 14 and 15, Art. 42, give for this condition:

$$M_3 = -l \frac{(2a + r) \Sigma P (1 - q^2) q}{4 (a + r) (b r^1 + 1) - r} \quad . \quad . \quad 284.$$

$$M_2 = -r l \frac{\frac{M_3}{l} + \Sigma P (1 - q^2) q}{2 (a + r)} \quad . \quad . \quad . \quad 285.$$

Eq. 57 of Art. 44 gives for this case:

$$\int_0^l M_b \, dx = \frac{l}{2} \left(M_2 + M_3 + \dot{\Sigma} P z (1 - q) \right) \quad . \quad 286.$$

For uniform loading, p extending from z_1 to z:

$$\dot{\Sigma} P z (1 - q) = \frac{p l^2}{2} \left(\frac{z^2}{l^2} - \frac{z_1^2}{l^2} - \frac{2}{3} \left(\frac{z^3}{l^3} - \frac{z_1^3}{l^3} \right) \right) \quad . \quad 287.$$

If the uniform load extends over the whole span, $z_1 = 0$ and $z = l$; hence:

$$\dot{\Sigma} P z (1 - q) = \frac{p l^2}{6} \quad . \quad . \quad . \quad . \quad . \quad 288.$$

As already indicated in connection with the dead and moving load in the side spans it is supposed that the ends of the stiffening-truss at G and K, Fig. 18, are held at their normal elevations by the pier structures to which they are attached.

Greatest Positive Reactions at Outer Ends of Spans l_1 and l_3, i.e., at G and K, Fig. 18.

The greatest positive or upward reaction at either G or K will occur at mean temperature when the entire spans l_1 and l_3 are covered with moving load and with no such load on the span l. If p' is the greatest moving load per linear unit on the two end spans l_1 and $l_3 = l_1$, and if s'_2 is the corresponding relief of suspender pull in the span l, then by Eq. 283:

$$s'_2 = \frac{p'}{p_1} w_2.$$

Eq. 269 then gives the maximum positive or upward reaction G, Fig. 18, *i.e.*, at the outer end of span l_1, still placing $l_2 = l$:

$$R_1 = \frac{p' l_1}{2} - \frac{l}{l_1} \frac{a^2 p' l_1 + r s'_2 l}{4 (2a + 3r)} \quad . \quad . \quad . \quad 289.$$

This reaction R_1 is obviously identical with the maximum upward or positive reaction R_4 at K, the outer end of the span l_3.

The end shear at the right-hand end of span l_1, *i.e.*, immediately at the left of D, will be:

$$S_D = R_1 - p^1 l_1 = -\frac{p^1 l_1}{2} - \frac{l}{l_1}\frac{a^2 p^1 l_1 + r s^1{}_2 l}{4(2a + 3r)} \quad . \quad 290.$$

Instead of the uniform load p' per linear foot covering the entire side spans, it may be more convenient to consider a system of uniform panel concentrations at each of the panel points in the spans l_1 and l_3. The preceding equations for the reactions R_1 and shear S_D are directly applicable to such a system of loading. It will only be necessary to write the amount of the total panel loads on each of the side spans for the expression $p' l_1$.

Shears and Bending Moments in Spans l_1 and l_3.

The general values of the shears and bending moments in the spans l_1 and l_3 may easily be written by the aid of the preceding analysis.

The general value of the shear S in the span l_1 may be expressed as follows, the value of the reaction R_1 being taken from Eq. 289:

$$S = R_1 - p^1 x = p^1\left(\frac{l_1}{2} - x\right) - \frac{l}{l_1}\frac{a^2 p^1 l_1 + r s^1{}_2 l}{4(2a + 3r)} \quad . \quad 291.$$

When $x = \frac{l_1}{2}$ the value of s in Eq. 291 becomes identical with the maximum upward reaction given by Eq. 289.

Again if $x = l_1$ the value of S becomes identical with the value S_D in Eq. 290. The point at which the shear $S = 0$ and at which the maximum bending moment is found may be at once determined from Eq. 291 by simply making $S = 0$. That operation will give for the point of greatest bending:

$$x = \frac{l_1}{2} - \frac{l}{l_1}\frac{a^2 l_1 + r l \frac{s^1{}_2}{p^1}}{4(2a + 3r)} \quad . \quad . \quad . \quad . \quad 292.$$

The general value of the bending moment in the span l_1, when there is no moving load in the span $l_2 = l$, becomes, after taking the value of R_1 from Eq. 289:

$$M = R_1 x - \frac{p' x^2}{2} = \frac{p' x}{2}(l_1 - x) - x\frac{l}{l_1}\frac{a^2 p' l_1 + r s'_2 l}{4(2a + 3r)} \quad . \quad 293.$$

If the value of x be computed from Eq. 292 and inserted in Eq. 293, the maximum positive bending moment in the span will result, and this will be an important quantity to be determined in an actual structure.

If $x = l_1$, Eq. 293 shows that M becomes equal to M_3 given by Eq. 268.

These few simple equations give the greatest values for the shears and bending moments at any point in either span l_1 or l_3. If the values for span l_3 are under consideration it is only necessary to change l_1 to l_3.

The maximum shears and bending moments are found by covering both of the side spans with the greatest prescribed moving loads. It is not necessary therefore to give values of shears and bending moments for the moving load covering one of the side spans only, although such values may readily be written.

Stresses in Spans l_1 and l_3, Due to Moving Load in Main Span—Greatest Negative Reactions.

Let M'_2 and M'_3 be the actual bending moments in the stiffening-truss at D and F for those positions of moving load which give them their greatest (negative) values. The greatest downward or negative reactions will then be at G and K, Fig. 18:

$$R'_1 = \frac{M'_2}{l_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 294.$$

And at K:

$$R'_3 = \frac{M'_3}{l_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 295.$$

After these two greatest negative reactions are known all the

corresponding stresses in the two spans l_1 and l_3 can at once be found by simple procedures, and tabulated.

The three tabulated sets of stresses due to:

Dead load in spans l_1 and l_3;

Moving load in spans l_1 and l_3, producing stresses in spans l_1, l_2, and l_3;

Moving load in span l_2 only, producing stresses in the stiffening-truss proper and in spans l_1 and l_3,

must then be combined so as to determine the greatest possible stresses in all members of the trusses.

The greatest amount of anchorage required at G and K, Fig. 18, will be given by Eqs. 294 and 295. If the spans l_1 and l_3 are relatively long, their dead load or own weight may be sufficient to induce no negative or downward reactions at G and K.

CHAPTER V.

Theory of the Straight Stiffening-Truss Based on the Method of Deflections.

Art. 63.—General Considerations.

In the treatment of the stiffened suspension bridge by the method of deflections the structure is assumed, as in other methods, to be so erected and adjusted that at a given or mean temperature the entire dead load hangs from the cable without producing any bending in the stiffening-truss. The constant action of the dead load, therefore, produces a fixed or unvarying stretching of the cable and suspenders superposed on which is the varying stretch of the same members due to the correspondingly varying application of moving load and in connection with which the moving load flexure or bending of the stiffening-truss occurs, the latter being subjected to no bending by the dead load or own weight.

In the following analysis, therefore, the elongations of the cable, the stretching of the suspenders, and the deflection of the stiffening-truss are those only due to the action of the moving load. On the other hand, however, the internal work done in producing the moving load deformations of the same members is performed by both moving and dead load.

The plan of treatment is first to find a general expression for the moving load deflections corresponding to the actual bending moments in the stiffening-truss. Inasmuch as the cable, the suspenders, and the stiffening-truss are united as members of a single structure, the combined deflection of the cable and the stretch of the suspenders at any point in the span must be the same as the deflection of the stiffening-truss at the same point. The work performed, therefore, by the dead and moving load acting through the vertical displacement caused by the moving

load must be equal to the total internal work expended in stretching the cable and suspenders and in deflecting, or bending, the stiffening-trusses throughout the span.

The moving load must be divided into two parts, one of which, usually much the smaller part, produces the deflection of the stiffening-truss and is carried by that truss to the two abutments or piers at the ends of the span as an ordinary continuous or non-continuous truss. The other and larger part of that load produces a uniform pull on the suspenders resulting in the cable stress due to the moving load. That part of the moving load resting upon the stiffening-truss as an ordinary continuous or non-continuous bridge structure obviously performs all the work expended in bending the stiffening-truss. On the other hand, the remaining part of the moving load which is carried by the cable performs work through its vertical displacement expended in stretching the suspenders and the cable.

When the moving load stretches the suspenders and the cable the dead load performs work over its vertical displacement. If then the work performed by the dead load and that part of the moving load carried to the cable, acting through their respective displacements, be equaled to the work expended in stretching the suspenders and cable, an expression will result from which the horizontal component of cable stress may be determined. These various procedures will be set forth and completed in the following analysis.

Similarly the work performed (supposedly by a uniform load over the entire span) in producing the displacement of the different parts of the structure through a change of temperature must be equal to the work expended in producing the corresponding deformations of the cable, suspenders, and stiffening-truss. The uniform load over the entire span which would produce the same cable and suspender stretch and the same stiffening-truss deflection as the change of temperature may be called the temperature or thermal load.

Art. 64.—Deflection of Truss and Cable-Stretch of Suspenders.

The following analysis requires the use of the notation given below, as well as more which will be set forth later:

H'' = Horizontal component of cable stress due to dead load.

H' = Horizontal component of cable stress due to moving load.

$H = H' + H''$.

P = Single moving load at distance z from one end of span.

w_0 = Total dead load per foot of span.

w = Total dead, moving, and temperature load per foot of span.

w'_1 = Total dead, moving, and temperature load, except the cable, per foot of span.

a = Panel length or horizontal distance between two consecutive suspenders.

$s_0 = w_0 - g_0$ = Dead load on suspenders, including their own weight per foot of span.

s' = Moving load and temperature pull on suspenders per linear foot of span.

w_1 = Moving load per foot of span for the length or part L_1. For moving load over entire span $L_1 = l$, and for that condition w_1 may represent temperature load.

g_0 = Weight of cable per linear foot at lowest point or vertex.

M_b = Bending moment at any point due to moving load in the stiffening-truss as an ordinary continuous or non-continuous truss unconnected with the cables.

M = Actual moving load bending moment at any point in the stiffening-truss.

S = Actual moving load shear at any point in the stiffening-truss.

γ = The specific weight of the cables per cubic inch.

u' = Deflection at any point of the cable due to moving load stresses.

v = Length of suspender at lowest point of the cable.

Δ = Stretch of whole length of a suspender at any point due to moving load stresses.

u = Deflection at any point of the stiffening-truss due to moving load stresses.

t = Allowed or assumed intensity of stress in the suspenders, due to moving load only.

T = Variation of temperature from the mean.

δ = Thermal change per unit of length for each degree of temperature.

L = Length in feet of cable in span l or l_2.

L' or L''' = Length in feet of cable in span l_1 or l_3.

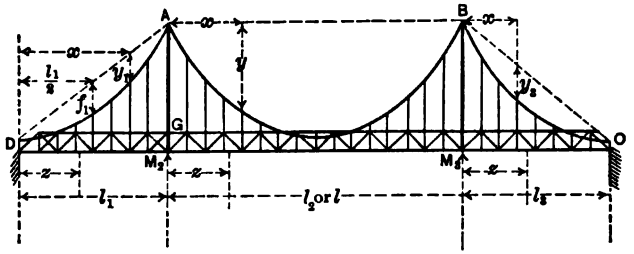


FIG. 1.

If the origin of co-ordinates be taken at the top of a tower, as shown in Fig. 1, the relation between the dead and moving load moments, the actual bending moments in the stiffening-truss, and the horizontal components of cable stress, H' and H'', will be:

$$\frac{w_0}{2} x (l - x) + M_b - M = (H' + H'') (y + u') \quad . \quad . \quad 1.$$

Hence, as $\frac{w_0}{2} x (l - x) = H'' (y + u')$:

$$M = M_b - H' (y + u') \quad . \quad . \quad . \quad . \quad . \quad 2.$$

In accordance with the notation:

$$u' = u - \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 3.$$

As the curve of the cable is a parabola, and as the origin of co-ordinates is at the top of a tower (Fig. 1):

$$y = \frac{4f}{l^2} x (l - x) \quad . \quad . \quad . \quad . \quad . \quad . \quad 4.$$

The ordinate y includes the deflection of the cable due to the dead load.

If v is the length of suspender at the apex of the cable, the stretch Δ of any suspender at the distance x from the end of the span due to moving load only will be:

$$\Delta = \frac{t}{E}(f - y + v) = \frac{t}{E}\left(f - \frac{4fx}{l^2}(l - x) + v\right) \quad . \quad 5.$$

This expression changes slightly for a side span l_1, Fig. 1, as follows:

$$\Delta_1 = \frac{t}{E}\left(v + \frac{f}{l_1} x - \frac{4f_1 x}{l_1^2}(l_1 - x)\right) \quad . \quad . \quad . \quad 5a.$$

The only difference between Eqs. 5 and 5a is that $\frac{f}{l_1} x$ takes the place of f in Eq. 5.

Substituting from Eqs. 3, 4, and 5 in Eq. 2:

$$-M = -M_b + H' \frac{4f}{l^2} x (l - x) + H' u - H' \frac{t}{E}(f + v)$$

$$+ H' \frac{t}{E} \frac{4f}{l^2} x (l - x) \quad . \quad . \quad . \quad . \quad . \quad 6.$$

In order to make this equation perfectly general M_b must be so written as to apply to a continuous stiffening-truss carrying a load P at any point in the span distant z from the end of the span. As has been done before, let $q = \frac{z}{l}$.

Then if the bending moment in a simple non-continuous span carrying a single load P at a point indicated by z be represented by M_0, and if P' be written for P in the manner indicated:

For $x \leq z$; $M_0 = P(1 - q)x = (P' - Pq)x \quad . \quad . \quad . \quad 7.$

For $x \geq z$; $M_0 = P\,q\,(l - x) = M_l - P\,q\,x$ . . . . 8.

Remembering that $P' = P$ when $M_l = 0$, *i.e.*, when $x \leq z$, and that $P' = 0$ when $M_l = P\,q\,l$, *i.e.*, when $x \geq z$, there may be written the expression for both cases:

$$M_0 = M_l + (P' - P\,q)\,x \text{ . . . . . } 9.$$

By referring forward to Eqs. 66, 67, and 68 of Art. 74, the general value of the moment M_b for stiffening-trusses continuous over three spans may be found so expressed as to be written:

$$M_b = M'_2 + M_l + \left(\frac{M_2}{l} + P' - P\,q\right)x \text{ . . . } 10.$$

For span l_1: $M'_2 = 0$ and $\frac{M_2}{l} = \frac{M_2}{l_1}$.

For span l_2: $M'_2 = M_2$ and $\frac{M_2}{l} = \frac{M_3 - M_2}{l_2}$.

For span l_3: $M'_2 = M_3$ and $\frac{M_2}{l} = -\frac{M_3}{l_3}$.

Using the familiar equation from the common theory of flexure:

$$E\,I\,\frac{d^2u}{dx^2} = -M;$$

and placing $\frac{H'}{E\,I} = c^2$, then by the aid of Eqs. 6 and 10, and after abbreviating by placing $N = \frac{M_2}{l} + P' - P\,q$, there will be found:

$$\frac{d^2u}{dx^2} - c_2\,u = -\frac{M'_2 + M_l}{E\,I} - c_2\frac{t}{E}(f+v) - \left(\frac{N}{E\,I} - c^2\frac{4f}{l}(1+\frac{t}{E})\right)x$$

$$- c^2\frac{4f}{l^2}\left(1 + \frac{t}{E}\right)x^2 \text{ . . . . . } 11.$$

Or:

$$\frac{d^2u}{dx^2} - c^2\,u = k + k'\,x + k''\,x^2.$$

The integral of this equation is:

$$u = C' e^{cx} + C'' e^{-cx} - \frac{1}{c^4}(2k'' + kc^2 + k'c^2x + k''c^2x^2) \quad . \quad 11a.$$

C' and C'' are constants of integration, and the limits of integration include only a part of the stiffening-truss for which the moment of inertia of section, I, is constant.

Substituting from Eq. 11 in Eq. 11a the quantities indicated by k, k', and k'':

$$*u = C'e^{cx} + C''e^{-cx} + \frac{M_b}{H'} + \frac{t}{E}(f+v) + \frac{4f}{l^2}\left(1+\frac{t}{E}\right)\left(\frac{2}{c^2} + x^2 - lx\right) \quad 12.$$

In this equation M_b, as shown by Eq. 3, is the moving load bending moment in a simply supported ordinary, continuous or non-continuous truss at the point x.

Eq. 12 remains unchanged for a side span, as l_1, after introducing the subscripts except, as indicated in Eq. 5a, $\frac{t}{E}\frac{f}{l_1}x$ must take the place of $\frac{t}{E}f$ in the fourth term of the second member

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*This expression for the deflection is not complete as the deflection due to the shear either in a solid plate web or in the web members of a truss is neglected. Terms representing the deflection due to shear may easily be added to the second member of Eq. 12, by using the expression for the shear in a solid plate web as follows:

$$\frac{d^2}{2IG}\int S\,dx;\ d \text{ being the half depth of web.}$$

The value of the shear S may be found in any convenient manner. Eq. 15 gives a value.

Or, again, the shear S multiplied by the secant of inclination of a truss web member to a vertical line will give the direct stress in that member. The deflection due to the tension and compression in such members throughout the entire span may then be found by the usual and familiar procedures.

The shearing deflection thus found must be added to u in Eq. 12, and used in the subsequent formulæ wherever u appears.

These procedures complicate but little the formulæ to be found, but the influence of shear on the deflection is so small that the error due to this omission is immaterial.

of that equation. This small change will appear in subsequent formulæ where $\int u_1 \, dx$ is to be found.

The preceding analysis by which Eq. 12 is established is perfectly general and is readily made applicable to uniform loading, although in form a single load P only is used. It will be shown later with what facility that equation and the consequent results may be applied either to any number of single loads or to any continuous loading, uniform or otherwise. If a special demonstration with uniform loading had been employed in establishing Eq. 12, the second member of that equation would stand precisely as it is, except that the term $-\frac{w_1}{c^2 H'}$ would be added.

Art. 65.—Bending Moment and Shear.

The actual bending moment in the stiffening-truss now becomes by Eq. 2:

$$M = M_b - H' (u - \Delta + y) \quad . \quad . \quad . \quad . \quad 13.$$

By the aid of Eqs. 4 and 12, this equation takes the form:

$$M = - H' (C' e^{cx} + C'' e^{-cx}) - E I \frac{8f}{l^2}\left(1 + \frac{t}{E}\right) \quad . \quad 14.$$

The shear at any point x in the stiffening-truss is:

$$S = \frac{dM}{dx} = - H' c (C' e^{cx} - C'' e^{-cx}) \quad . \quad . \quad . \quad 15.$$

This equation shows that the shear in the stiffening-truss is not dependent upon the stretching of the suspenders.

The rate of variation of the shear, *i.e.*, the variation of the shear per unit of length of truss, is:

$$\frac{dS}{dx} = \frac{d^2 M}{dx^2} = - H' c^2 (C' e^{cx} + C'' e^{-cx}) \quad . \quad . \quad 15a.$$

Eqs. 14, 15, and 15*a* apply to a side span l_1, with no other change than the introduction of subscripts.

The known values of the moments M and shears S at given points will aid in the determination of the constants C' and C''.

This may be illustrated by the simple case of a non-continuous stiffening-truss of span l and with a uniform moment of inertia I, for which the moment $M = 0$ when either $x = 0$ or $x = l$. Hence Eq. 14 will then yield the two equations of condition:

$$C' + C'' = -\frac{EI}{H'}\frac{8f}{l^2}\left(1 + \frac{t}{E}\right) \quad . \quad . \quad . \quad 15b.$$

$$C' e^{cl} + C'' e^{-cl} = -\frac{EI}{H'}\frac{8f}{l^2}\left(1 + \frac{t}{E}\right) \quad . \quad . \quad . \quad 15c.$$

Multiplying Eq. 15c by e^{cl}, then substituting for C'' from Eq. 15b, a simple reduction will give:

$$C' = -\frac{1}{1 + e^{cl}}\frac{EI}{H'}\frac{8f}{l^2}\left(1 + \frac{t}{E}\right) \quad . \quad . \quad . \quad 15d.$$

Hence, by the aid of Eq. 15b:

$$C'' = -\frac{1}{1 + e^{-cl}}\frac{EI}{H'}\frac{8f}{l^2}\left(1 + \frac{t}{E}\right) \quad . \quad . \quad . \quad 15e.$$

These quantities will be needed in making a practical application of this analysis to non-continuous stiffening-trusses.

$$\text{If } J = -H'\left(1 + \frac{t}{E}\right)\frac{8f}{c^2 l^2} = -EI\left(1 + \frac{t}{E}\right)\frac{8f}{l^2},$$

Eq. 14 may be written:

$$M = -H'(C' e^{cx} + C'' e^{-cx}) + J \quad . \quad . \quad . \quad 16.$$

Art. 66.—Pull on Suspenders.

The dead load pull on the suspenders per linear foot of span is obviously the total dead load except that of the cable, *i.e.*, $s_0 = w_0 - g_0$.

The moving load pull and the pull due to temperature load, *i.e.*, in the suspenders, are both represented in what follows by s' per linear foot. This will not be known until H' is determined. As s' must be used in the formulæ by which H' is found, it may be estimated with close approximation by the simple statical formulæ of Chapter I. After finding H', s' may be accurately computed, and if the approximate value should not prove to be

sufficiently close a revised solution for H' can be made; but this will seldom be necessary.

The tangent of inclination of the cable to a horizontal line is by Eq. 4:

$$\frac{dy}{dx} = \frac{4f}{l^2}(l - 2x) \quad . \quad . \quad . \quad . \quad . \quad 16a.$$

At the top of a tower $x = 0$, and:

$$\left(\frac{dy}{dx}\right)_0 = \frac{4f}{l}$$

When $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$.

In general, then:

$$H'\frac{dy}{dx} = \frac{4H'f}{l^2}(l - 2x) = s'\left(\frac{l}{2} - x\right)$$

$$\therefore s' = \frac{8H'f}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 16b.$$

If the centre deflection of the cable under moving load or temperature variation is $\pm u_c$:

$$s' = \frac{8H'(f \pm u_c)}{l^2} \quad . \quad . \quad . \quad . \quad . \quad 16b'.$$

If a is the panel length between two consecutive suspenders, the moving load or temperature pull on one suspender will be:

$$P' = a\,s' \quad . \quad . \quad . \quad . \quad . \quad . \quad 16c.$$

The total dead and moving load pull on a suspender will then be:

$$P = a\,(s_0 + s') \quad . \quad . \quad . \quad . \quad . \quad 16d.$$

If ΣP or $w_1 L_1$ is the total moving load on the span, while S_0 and S'_0 are the two end shears in the stiffening-truss (*i.e.*, at the two extremities of the span), that part of the total moving load carried by the stiffening-truss, as a truss only, is:

$$\Sigma P \text{ or } w_1 L_1 - s'\,l = S_0 + S'_0 \quad . \quad . \quad . \quad 16e.$$

Thus when once the horizontal component H' of the moving load cable stress is known, all moments and stresses in the entire structure can easily be computed.

Art. 67.—Constants of Integration C' and C''.

Let $o - a_1$, $a_1 - a_2$, $a_2 - a_3$, etc., be the parts of the span over each of which the moment of inertia I is constant, those constant values of I being I_1, I_2, I_3, etc. Then the constants of integration, varying with c which contains I the moment of inertia, will take the following sets of values for the different parts of the span indicated in the various moments and shears at the points $x = o$, $x = a_1$, $x = a_2$, $x = a_3$, etc.

$$M_0 = -H'(C'_1 + C''_1) + J \quad . \quad . \quad . \quad . \quad . \quad 17.$$

$$M_1 = -H'(C'_1 e^{ca_1} + C''_1 e^{-ca_1}) + J = -H'(C'_2 e^{ca_1} + C''_2 e^{-ca_1}) + J \quad . \quad . \quad . \quad 18.$$

$$M_2 = -H'(C'_2 e^{ca_2} + C''_2 e^{-ca_2}) + J = -H'(C'_3 e^{ca_2} + C''_3 e^{-ca_2}) + J \quad . \quad . \quad . \quad 19.$$

etc. = etc., etc., etc.

Other similar sets of equations may be written until the entire span l is covered by them.

If M_3 is at the opposite end of the span from M_0, $a_3 = l$, and:

$$M_l = -H'(C'_3 e^{cl} + C''_3 e^{-cl}) + J \quad . \quad . \quad . \quad 20.$$

The number of these equations will be one more than the number of sections in the span l, while the number of the constants C'_1, C''_1, C'_2, C''_2, etc., will be double the number of the sections. In general, then, other equations of condition must be found, which can easily be done, as shown by the following illustration.

If there are three sections in the span, there will be six constants to be determined by the aid of the four equations above. If the trusses are non-continuous the end moments are zero, and $M_0 = M_l = o$. This makes all of the equations 17, 18, 19, and 20 available. Again, u, Eq. 12, must be zero for both $x = o$

and $x = l$, since the end deflections of the truss are zero. These two equations are:

$$\text{For } x = 0\text{: } C'_1 + C''_1 + \frac{t}{E}(f+v) + \frac{8f}{c^2 l^2}\left(\frac{t}{E}+1\right) = 0 \quad . \quad 21.$$

$$\text{For } x = l\text{: } C'_3 e^{cl} + C''_3 e^{-cl} + \frac{t}{E}(f+v) + \frac{8f}{c^2 l^2}\left(\frac{t}{E}+1\right) = 0 \quad . \quad 22.$$

Evidently with non-continuous trusses $M_0 = 0$ when $x = 0$ and $x = l$. Also, in general, c is the same for both ends of the span l. Hence the parts of Eqs. 21 and 22 independent of the constants sought are identical. The six equations 17 to 22, both inclusive, will then be sufficient to determine the six constants of integration.

If the stiffening-trusses are continuous, M_0 and M_l will, in general, not be known; but Eqs. 21 and 22 will be available after giving M_b (Eq. 10) its proper value (Eqs. 62 and 63) for each end of the span and adding it to the first member of each of those equations. For a side span, l_1, $M_b = 0$ for the anchorage end of the span. Also if the truss is held horizontal or "fixed" at each tower $\frac{du}{dx} = 0$, for $x = 0$ and $x = l$.

For $x = 0$:

$$\frac{du}{dx} = c(C'_1 - C''_1) + \frac{1}{H'}\left(\frac{dM_b}{dx}\right)_{x=0} - \frac{4f}{l}\left(\frac{t}{E}+1\right) = 0 \quad . \quad 23.$$

For $x = l$:

$$\frac{du}{dx} = c(C'_3 e^{cl} - C''_3 e^{-cl} + \frac{1}{H'}\left(\frac{dM_b}{dx}\right)_{x=l} + \frac{4f}{l}\left(\frac{t}{E}+1\right) = 0 \quad . \quad 24.$$

The six equations 18, 19, 21, and 22 (changed as indicated), 23, and 24, will enable the six constants of integration to be determined.

Finally, there will be a pair of equal values of $\frac{du}{dx}$ with different pairs of constants at each point between two adjacent sections, a_1, a_2, etc., thus forming an equation of condition at each such point, similarly to the formation of Eqs. 18 and 19.

Hence, in all cases there will be a sufficient number of equations to find the constants of integration.

After the horizontal components H' and H'' of the cable stresses are known, therefore, the deflection of the stiffening-truss, the bending moments M and shears S in that truss may all be determined.

If the moment of inertia I and the moving load per linear foot w_1 are each constant throughout the span, then will the shear $S = 0$ for the centre of the span, and Eq. 15 will give:

$$C'' = C' e^{d}.$$

In general, if the shear $S = 0$ at the point located by x, then by Eq. 15:

$$C'' = C' e^{2cx}.$$

Art. 68.—Effect of Stresses in the Cable Due to Moving or Temperature Load.

Stress in the cable may be developed by moving load on either or both of the side spans or on the centre span or by change of temperature, as will be obvious by referring to the conditions indicated by Fig. 1 of Art. 64. In any event the total length of cable between anchorages must be considered. Changes in the length of cable accompany corresponding changes of cable stresses and of stresses in the suspenders. The loads on the latter, vertical in direction, may be considered as acting at the lower ends of the suspenders where they join the stiffening-truss, and they may be treated as forces external to the suspender and cable systems. The work performed by these external forces over the vertical deflections, *i.e.*, their paths, must be equal to the energy expended in stretching the cable and suspenders.

Art. 69.—Energy Expended in Stretching Suspenders.

Let s' = Moving load and temperature pull on suspenders per linear foot of span due to moving load ΣP or $w_1 L_1$ and change T of temperature.

Let a = Panel length in feet between two consecutive suspenders.

Let δ = Coefficient of thermal expansion per degree Fahr.

The total dead and moving load and the thermal pull (if any), on one suspender, will then be:

$$a\, s_0 \text{ (dead)} + a\, s' \text{ (moving and thermal)}.$$

Hence the mean load acting at the lower end of each suspender over the deflection u is $a\,(s_0 + \frac{1}{2}\, s')$, or for the distance $d\,x$ the load will be $(s_0 + \frac{1}{2}\, s')\, d\,x$. The change of length in the suspender due to the moving load unit stress t is given by Eq. 5. The work expended in stretching the suspenders of one span is, therefore:

$$\int_0^l \Big(s_0 + \tfrac{1}{2}\, s' \Big) \frac{t}{E} \Big(f - \frac{4\,f}{l^2}\, x\,(l - x) + v \Big) d\,x = \Big(s_0 + \tfrac{1}{2}\, s' \Big) \frac{t}{E}\, l\,(v + \tfrac{1}{3}\, f) \quad . \quad . \quad . \quad . \quad 25.$$

Inasmuch as the pull on the suspenders is always uniform throughout the span, Eq. 25 is directly applicable to any condition of loading whatever. It is only necessary to determine the moving load pull s' on the suspenders due to the loading assumed. The same equation is also directly applicable to the side span l_1 by using the mid ordinate f_1 and the span length l_1, if the side spans carry loading, *i.e.*, if they have suspenders.

The work expended in stretching the suspenders of a side span may be written as follows by the aid of Eq. 5*a*:

$$\int_0^{l_1} (s_0 + \tfrac{1}{2}\, s')\, \Delta_1\, d\,x = (s_0 + \tfrac{1}{2}\, s') \frac{t}{E}\, l_1 \Big(\frac{f}{2} + v - \tfrac{2}{3}\, f_1 \Big) \quad . \quad 25a.$$

In the usual case of each of the side spans being one-half the main span, $l_1 = \frac{l}{2}$ and $f_1 = \frac{f}{4}$. Hence $\frac{f}{2} + v - \frac{1}{2}\, f_1 = v + \frac{f}{3}$ as in Eq. 25. In such a case, therefore, Eq. 25*a* shows that the work done in stretching the suspenders of a side span is one-half that similarly expended in the centre or main span.

If there are no suspenders in the side spans, no such expressions as Eq. 25*a* will be required for those spans.

Art. 70.—Energy Expended in Stretching the Cable.

Let F be the area of cross-section of the cable if it be of uniform section, as for a wire cable, or the area of cross-section at its lowest point if that area varies as the cable stress, as for an eye-bar cable. If L is the length of cable between the tops of towers and H' or H'', respectively, the horizontal component of the moving or dead load cable stress, as heretofore, the moving load cable stress at any point x will be $H'\frac{dL}{dx}$. The moving load stretch of dL will be $\frac{H'}{FE}\frac{dL}{dx}dL$. The dead load cable stress performs work over that path or stretch. The mean live load cable stress also performing work over the same path is one-half of that given above. Hence the total mean stress is $(H'' + \frac{1}{2}H')\frac{dL}{dx}$, and the energy expended on the small part dL of the cable will be the product of the two quantities:

$$\frac{H'(H'' + \frac{1}{2}H')}{FE}\frac{dL^2}{dx^2} = dW.$$

The total work is, therefore:

$$W = \frac{H'(H'' + \frac{1}{2}H')}{E}\int_0^l \frac{dL^2}{F\,dx^2} = \frac{H'(H'' + \frac{1}{2}H')}{E}\int_0^l \frac{dL^2}{F\,dx^2}dx \quad . \quad 26.$$

If the cable is of uniform section throughout its length, F is constant and Eq. 26 becomes:

$$W = \frac{H'(H'' + \frac{1}{2}H')}{FE}\int_0^l \frac{dL^2}{dx^2}dx \quad . \quad . \quad . \quad 27.$$

If the section of the cable varies as the cable stress the general value of that section is $F\frac{dL}{dx}$. Eq. 26 then becomes:

$$W = \frac{H'(H'' + \frac{1}{2}H')}{FE}\int_0^l \frac{dL^2}{dx^2}dx \quad . \quad . \quad . \quad 28.$$

Eq. 27 or Eq. 28 is applicable without change of form to each of the side spans when suspenders are found in them. If the ordinary case is assumed in which the cable in each side span may be taken, the same as half the cable in the main span, the work of stretching a side span cable will be half that expended in stretching the entire cable in the centre span.

When there are no suspenders in the side spans the cables of the latter, or the "backstays," are essentially straight, and the area of cross-section is constant. For this case, if L' is the length of cable or backstay and φ the angle which it makes with a horizontal line, so that $sec\ \varphi = \frac{d\,L'}{d\,x'}$, Eq. 27 becomes:

$$W_1 = \frac{H'\,(H'' + \tfrac{1}{2}\,H')}{F_1\,E}\,sec^3\,\varphi\,l_1 = \frac{H'\,(H'' + \tfrac{1}{2}\,H')}{F_1\,E}\,sec^2\,\varphi\,L' \quad . \quad 29.$$

If the structure is composed of one main span and two backstay spans, the total work performed in stretching the cable by the moving load will be given by

$$W + 2\,W_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad 29a.$$

Art. 71.—Energy Expended Due to Thermal Changes in Cable.

Stress conditions due to changes in temperature may occur with the structure either loaded with live load or without that load; they result from the change of pull in the suspenders consequent upon the thermal changes in length of the cable and suspenders. Those changes are equivalent to the action of the corresponding thermal load uniform over the entire span or spans and included in w_1. The corresponding change in cable stress is included in the value of $H'\frac{d\,L}{d\,x}$, the horizontal component H' representing the effect of live load and thermal or other influence.

The thermal change in length of the element $d\,L$ of the cable for a change of T degrees in temperature is $T\,\delta\,d\,L$, and the mean cable stress at the same point is $(H'' + \tfrac{1}{2}\,H')\frac{d\,L}{d\,x}$, as

has already been stated. The work performed by that stress throughout the entire cable will, therefore, be:

$$T\,\delta\,(H'' + \tfrac{1}{2}\,H')\int_0^L \frac{dL}{dx}\,dL = T\,\delta\,(H'' + \tfrac{1}{2}\,H')\int_0^l \frac{d\,L^2}{d\,x^2}\,dx\,. \quad 30.$$

The length L is the length of cable between the tops of towers. If the stiffening-trusses extend over the side spans l_1 and l_3 and are carried by suspenders, Eq. 30 is to be applied to those spans simply by introducing the proper subscripts. For the ordinary case of the side span being essentially a duplicate of half the centre span, the thermal work done on the cable of the side span will be half that similarly expended in the centre or main span. If backstays only are in the side spans, as they are straight, $\frac{dL}{dx} = sec\,\varphi$, and Eq. 30 will take the form for each side span l_1 or l_3:

$$T\,\delta\,(H'' + \tfrac{1}{2}\,H')\,sec^2\,\varphi\,l_1 = T\,\delta\,(H'' + \tfrac{1}{2}\,H')\,sec\,\varphi\,L'\,. \quad 31.$$

As in Eq. 29, L' is the length of backstay from the top of the tower to the anchorage.

Art. 72.—Energy Expended Due to Thermal Changes in Suspenders.

The thermal work done in the suspenders can be found by simply placing $T\,\delta$ for $\frac{t}{E}$ in Eq. 25:

$$(s_0 + \tfrac{1}{2}\,s')\,T\,\delta\,(v + \tfrac{1}{3}\,f)\,l \quad . \quad . \quad . \quad . \quad 32.$$

The same substitution of $T\,\delta$ for $\frac{t}{E}$ in Eq. 25*a* will give the thermal work for each side span:

$$(s_0 + \tfrac{1}{2}\,s')\,T\,\delta\,(v + \frac{f}{2} - \tfrac{2}{3}\,f_1)\,l_1 \quad . \quad . \quad . \quad 33.$$

For the side span l_3, the subscript 3 is to be used instead of 1 in Eq. 33. As shown in connection with Eq. 25*a*, if $l_1 = \frac{1}{2}\,l$, $v + \frac{f}{2} - \frac{2}{3}\,f_1 = v + \frac{f}{3}$.

Art. 73.—Work Performed Through Vertical Deflection.

The total dead, thermal, and moving load w is vertically displaced through the deflection u', Eq. 3, and in addition thereto the same total load less the weight of the cable is displaced vertically through the deflection Δ due to the stretch of the suspenders and to the change in length of the same members caused by thermal variations, as shown by Eq. 5. Strictly speaking, the weight of the suspenders is displaced by their stretch and thermal change in length only $\frac{1}{2}\Delta$, but their weight is so small relatively that it may well be placed with the dead load of that part of the structure attached to their lower extremities.

Inasmuch as the moving load is distributed uniformly to the cable through the suspenders (as is the thermal load), it may be considered uniformly distributed* over the span for the present purpose, and the same distribution will be assumed for the total dead load.

The work performed by the moving, thermal, and dead load will therefore be, remembering that $u = u' + \Delta$:

$$Z = \int_0^l w\,dx\,u - \int_0^l g_0\,dL\,\Delta = \int_0^l w\,u\,dx - g_0\int^l \Delta\,\frac{dL}{dx}dx. \quad 34.$$

An exactly similar expression may be written for a side span:

$$Z_1 = \int_0^{l_1} w\,u_1\,dx - g_0\int_0^{l_1} \Delta_1\frac{dL^1}{dx}\,dx \quad . \quad . \quad . \quad 35.$$

If the cross-section of the cable is not constant, but proportional to its total stress, g_0 will be its weight per linear foot at the lowest point or apex of the curve, and in Eq. 34 there should be

*This assumption is necessary for a simple treatment of the work performed by the moving load through vertical deflection, but it can scarcely be considered satisfactory. For uniform loading over a short distance such as a panel length of truss, or for a single concentrated load at a panel point, the assumption of uniform distribution over the entire span is certainly arbitrary and not likely to lead to a closely approximate expression for the work actually performed by either of such loadings.

taken $g_0 \frac{dL}{dx}$ for g_0. Hence in the last term of the last member of each of Eqs. 34 and 35 there should be written:

$$\frac{dL^2}{dx^2} \text{ for } \frac{dL}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 36.$$

The work performed in deflecting the stiffening-truss is done by that part of the moving load carried by it as an ordinary truss to the piers or abutment at the ends of the span. This work is to be omitted in what follows from the total work represented by Eqs. 34 and 35, because that part only of the work performed through vertical deflection and expended in stretching the suspenders and cable is needed. That omission is made by substituting $w_0 + \frac{1}{2} s'$ for w in Eqs. 34 and 35.

Since all load on the structure is considered to be uniformly distributed, if w_0 is the total dead load, including the weight of the cable, there may be written:

$$(w_0 + s')\frac{l^2}{8} = (f \pm u_c)(H'' + H') \quad . \quad . \quad . \quad 37.$$

In this equation u_c is the deflection, due to moving load on the cable, at the centre of span from its mean position under the conditions assumed and approximately determined by any convenient process. The quantity f includes the deflection due to dead load.

An average value for w, *i.e.*, $w_0 + \frac{1}{2} s'$, may be written as follows:

$$w_0 + \tfrac{1}{2} s' = \frac{8(f \pm u_c)}{l^2}(H'' + \tfrac{1}{2} H') \quad . \quad . \quad . \quad 38.$$

The first member of Eq. 38 is to take the place of w in Eqs. 34 and 35, and it is to be given the value shown by the second member of Eq. 38. The amounts of the work performed in the main and side spans will, by the aid of Eqs. 34 and 35, then become:

$$Z = \frac{8(f \pm u_c)}{l^2}(H'' + \tfrac{1}{2} H') \int_0^l u\, dx - g_0 \int_0^l \Delta \frac{dL}{dx}\, dx \quad . \quad 39.$$

$$Z_1 = \frac{8\,(f_1 \pm u_{c1})^*}{l_1^2}\,(H'' + \tfrac{1}{2}H')\int_0^{l_1} u_1\,dx - g_0\int_0^{l_1}\Delta_1\frac{dL'}{dx}\,dx. \quad 40.$$

Frequently the deflection u_c may be considered zero without essential error, but it can be computed approximately with little trouble wherever desired.

Art. 74.—General Value of H'.

The work performed by the dead, moving, and thermal loads in vertical displacement, except that done in deflecting the stiffening-truss, must be equal to the internal work performed in stretching the cable and suspenders by the moving loads and variations of temperature. Hence by the aid of Eqs. 39, 40, 26, 30, 25, 25a, 32, and 33, there may be written:

$$\frac{8\,(f \pm u_c)}{l^2}(H'' + \tfrac{1}{2}H')\Big(\int_0^l u\,dx + 2\int_0^{l_1} u_1\,dx\Big) - g_0\Big(\int_0^l \Delta\frac{dL}{dx}dx + 2\int_0^{l_1}\Delta_1\frac{dL'}{dx}\,dx\Big) = \frac{H'\,(H'' + \tfrac{1}{2}H')}{E}\Big(\int_0^l \frac{dL^2}{F\,dx^2}dx + 2\int_0^{l_1}\frac{dL'^2}{F_1\,dx^2}dx\Big) \pm T\delta\Big(H'' + \tfrac{1}{2}H'\Big)\Big(\int_0^l \frac{dL^2}{dx^2}dx + 2\int_0^{l_1}\frac{dL'^2}{dx^2}dx\Big) + (s_0 + \tfrac{1}{2}s^1)\Big(\frac{l}{E} \pm T\delta\Big)l\Big[v + \tfrac{1}{3}f + 2\frac{l_1}{l}\Big(\frac{f}{2} + v - \tfrac{2}{3}f_1\Big)\Big]. \quad 41.$$

If the cross-section of the cable varies as its total stress, for g_0 there is to be written $g_0\dfrac{dL}{dx}$, as is shown by expression 36, so that $\dfrac{dL^2}{dx^2}$ under each of the integrals in the second parenthesis of the first member of Eq. 41 will then take the place of $\dfrac{dL}{dx}$.

The integrals multiplied by 2 indicate that one integral is to be taken for each side span. If the side spans are of unequal lengths, the integrals will evidently be correspondingly unequal.

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*As the two parabolas have the same perameters:

$$\frac{f \pm u_c}{l^2} = \frac{f_1 \pm u_{c_1}}{l_1^2}.$$

If the cable is of constant cross-section, F will be constant and come out from under the integral signs. If the cross-section varies with the total stress, F will be the section at the lowest point and it must be displaced by $F \frac{dL}{dx}$.

In some cases it may be considered proper to neglect the stretching of the suspenders. Both Δ and Δ_1 will then be zero, as will also be the last term of the second member of Eq. 41.

Bridge with No Side Spans.

If the stiffening-trusses are found in the main span l, only, *i.e.*, if they do not extend into the side spans, there will be no suspenders in the latter. Hence $u_1 = 0$ and $\Delta_1 = 0$, while F_1 is constant. Also,

$$\frac{dL'}{dx} = sec\ \varphi;\ 2\int_0^{l_1}\left(\frac{dL'}{dx}\right)^3 dx = 2\ l_1\ sec^3\varphi;\ \text{and}\ 2\int_0^{l_1}\left(\frac{dL'}{dx}\right)^2 dx$$

$= 2\ l_1\ sec^2\varphi$; as indicated by Eq. 29. Finally the expression $2\frac{l_1}{l}\left(\frac{f}{2} + v - \frac{2}{3}f_1\right)$ in the last term of the second member of Eq. 41 will disappear. That equation will then take the form:

$$\frac{8(f \pm u_c)}{l^2}(H'' + \tfrac{1}{2}H')\int_0^l u\,dx - g_0\int_0^l \Delta\frac{dL}{dx}dx =$$

$$\frac{H'(H''+\frac{1}{2}H')}{E}\left(\int_0^l \frac{dL^3}{Fdx^3}dx + 2\ l_1\ sec^3\varphi\right) \pm T\delta\,(H'' + \tfrac{1}{2}H')$$

$$\left(\int_0^l\left(\frac{dL}{dx}\right)^2 dx + 2\ l_1\ sec^2\varphi\right) + (s_0 + \tfrac{1}{2}s')\left(\frac{t}{E} \pm T\delta\right)l\,(v+\tfrac{1}{3}f)\ .\ 42.$$

Values of Integrals in Eq. 41.

Six of the eight integrals in Eq. 41 involve the derivatives $\frac{dL}{dx}$ or $\frac{dL'}{dx}$, and as the curve of the loaded cable in all spans is the common parabola they are readily determined as follows.

For the purpose of taking the integrals throughout each

span it will be convenient to place the origin of co-ordinate at the apex of the curve (*i.e.*, at the centre of span) for the span l_2 or l, and at the corresponding point in the span l_1, *i.e.*, at the lower extremity of f_1 in Fig. 1, Art. 64.

For the main span l:

$$y = \frac{4f}{l^2}x^2; \text{ and } \frac{dy}{dx} = \frac{8f}{l^2}x \quad . \quad . \quad . \quad . \quad . \quad 43.$$

As $dL = (dy^2 + dx^2)^{\frac{1}{2}}$, it follows that:

$$\frac{dL}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = \left(1 + \frac{64f^2}{l^4}x^2\right)^{\frac{1}{2}} \quad . \quad . \quad 44.$$

The solution of Eq. 41 requires the values of $2\int_0^{\frac{l}{2}} \frac{dL}{dx}dx$, $2\int_0^{\frac{l}{2}} \frac{dL}{dx}\,x\,dx$ and $2\int_0^{\frac{l}{2}} \frac{dL}{dx}x^2dx$; also of $2\int_0^{\frac{l}{2}} \left(\frac{dL}{dx}\right)^3 dx$ and $2\int_0^{\frac{l}{2}} \left(\frac{dL}{dx}\right)^2 dx$. Precisely the same values are required for the side spans.

Using the third member of Eq. 44 for $\frac{dL}{dx}$:

$$2\int_0^{\frac{l}{2}} \frac{dL}{dx}dx = 2\int_0^{\frac{l}{2}} \left(1 + \frac{64f^2}{l^4}x^2\right)^{\frac{1}{2}} dx = \frac{l}{2}\left[\sqrt{1 + \frac{16f^2}{l^2}}\right.$$

$$\left. + \frac{l}{4f}\log\left(\frac{4f}{l} + \sqrt{1 + \frac{16f^2}{l^2}}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad 45.$$

Factoring the third member of Eq. 44:

$$2\int_0^{\frac{l}{2}} \frac{dL}{dx}\,x\,dx = \frac{16f}{l^2}\int_0^{\frac{l}{2}} \left(\frac{l^4}{64f^2} + x^2\right)^{\frac{1}{2}} x\,dx =$$

$$l^2\frac{l^2}{96f^2}\left\{\left(1 + \frac{16f^2}{l^2}\right)^{\frac{3}{2}} - 1\right\} \quad . \quad . \quad . \quad 46.$$

If $\left(1 + \frac{16f^2}{l^2}\right)^{\frac{3}{2}}$ be developed by the binomial formula and if the first three terms only of the development be used:

$$2\int_0^{\frac{l}{2}} \frac{dL}{dx}\,x\,dx = \frac{l^4}{96f^2}\left\{\left(1 + \frac{16f^2}{l^2}\right)^{\frac{3}{2}} - 1\right\} = \tfrac{1}{4}l^2 + f^2 \quad . \quad 47.$$

The error in this result is less than one-fifth of one per cent. for usual ratios of f to l; hence Eq. 47 may always be used.

Again:

$$2\int_0^{\frac{b}{2}} \frac{dL}{dx} x^2\, dx = l^3 \frac{l^2}{256 f^2}\left\{\left(1+\frac{16f^2}{l^2}\right)^{\frac{1}{2}}\left(\tfrac{1}{2}+\frac{16f^2}{l^2}\right) - \frac{l}{8f}\log\left(\frac{4f}{l}+\sqrt{1+\frac{16f^2}{l^2}}\right)\right\} \quad . \quad . \quad . \quad 48.$$

Proceeding in the same general way:

$$2\int_0^{\frac{l}{2}} \left(\frac{dL}{dx}\right)^2 dx = \frac{l}{4}\left\{\left(1+\frac{16f^2}{l^2}\right)^{\frac{1}{2}}\left(\frac{5}{2}+\frac{16f^2}{l^2}\right) + \tfrac{3}{8}\frac{l}{f}\log\left(\frac{4f}{l}+\sqrt{1+\frac{16f^2}{l^2}}\right)\right\} \quad . \quad . \quad . \quad 49.$$

The remaining integral is simple and easily found:

$$2\int^{\frac{l}{2}}\left(\frac{dL}{dx}\right)^2 dx = 2\int_0^{\frac{l}{2}}\left(1+\frac{64f^2}{l^4}x^2\right)dx = l\left(1+\frac{16f^2}{3l^2}\right) \quad 50.$$

All these expressions, Eqs. 45 to 50, both inclusive, are applicable to a side span by simply changing f to f_1, and $\frac{l}{2}$ to l_1. In the usual case where the sag of the side span cable below the top of the tower is approximately equal to f and $l_1 = \frac{l}{2}$, each of the preceding quantities for the main span may be taken as the sum of those for the two side spans.

Integrals $\int_0^l u\, dx$ *and* $\int_0^{l_1} u_1\, dx$.

These integrals are readily obtained by the aid of Eq. 12. From well-known integrals the two following expressions may be written:

$$\int_0^l C' e^{cx}\, dx = C'\int_0^l e^{cx}\, dx = \frac{C'}{c}\left(e^{cl}-1\right) \quad . \quad . \quad 51.$$

$$\int_0^l C'' e^{-cx}\, dx = C''\int_0^l e^{-cx}\, dx = \frac{C''}{c}\left(1-e^{-cl}\right) . \quad . \quad 52.$$

Precisely the same integrals are to be used for the side spans, but l is then to be displaced by l_1. Also C' and C'' will be determined for those spans.

The remaining integrals in these expressions are so simple that they need no further attention, except $\int M_b \, dx$ will be treated later.

Δ and Δ_1 for Eq. 41.

As already stated, Eqs. 45 to 50, both inclusive, are established for the origin of co-ordinates on the curve at the centre of the main span. Where those expressions are multiplied by Δ or Δ_1 in Eq. 41, the latter quantities may readily be written with the same origin of co-ordinates.

Under this condition the length of a suspender at the distance x from the centre of the main span will be:

$$y + v = \frac{4f}{l^2} x^2 + v \quad . \quad . \quad . \quad . \quad . \quad 53.$$

Hence the stretch of the suspenders will take the form:

$$\Delta = \frac{t}{E} \left(v + \frac{4f}{l^2} x^2 \right) \quad . \quad . \quad . \quad . \quad . \quad 54.$$

This is the value to be used in Eq. 41 for the term $\int \Delta \frac{dL}{dx} dx$.

If x and y, horizontal and vertical co-ordinates, be measured from the centre of each side span, but from an origin on the curve with y measured upward from a tangent to the curve at its apex, parallel to $A\,D$, Fig. 1, Art. 64, the equation of the curve will be:

$$y = \frac{4f_1}{l^2_1} x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 55.$$

The length of a suspender will be:

$$(\tfrac{1}{2} l_1 \pm x) \frac{f + v}{l_1} - f_1 + y = \frac{f + v}{2} \pm \frac{f + v}{l_1} x - f_1 + \frac{4f_1}{l^2_1} x^2 . \quad 56.$$

The stretch of the suspender will therefore be:

$$\Delta_1 = \frac{t}{E}\left(\frac{f+v}{2} - f_1 \pm \frac{f+v}{l_1}x + \frac{4f_1}{l_1^2}x^2\right) \quad . \quad . \quad 57.$$

This is to be used in the integral expression $\int \Delta_1 \frac{dL}{dx}\,dx$.

Value of H' from Eq. 41.

It will be observed that except for the factor l, l^2, or l^3 the second members of Eqs. 45 to 50 involve no variable except the ratio $\frac{f}{l}$ or $\frac{f_1}{l_1}$ which is nearly always found between $\frac{1}{8}$ and $\frac{1}{12}$. Hence let the second members of Eqs. 45, 46, 48, 49, and 50 be written as follows:

	For span l	For span l_1
Eq. 45	lA	$l_1 A_1$
" 46	l^2B	$l_1^2 B_1$
" 48	l^3C	$l_1^3 C_1$
" 49	lD	$l_1 D_1$
" 50	lG	$l_1 G_1$

The following tabular values for A, B, C, D, and G have been computed as shown for ratios of $\frac{f}{l}$ varying from .2 to .05, which represent limits somewhat greater and less than the ratios ordinarily employed in stiffened suspension bridge design. These values may be used for side spans as well as the main span in which suspenders and stiffening-trusses are found. They answer, therefore, for A_1, B_1, C_1, D_1, and G_1, as well as for the quantities without subscript.

$\frac{f}{l}$	A	B	C
.2	1.098,223	.286,514	.098,437
.175	1.076,473	.278,412	.094,671
.15	1.056,934	.270,949	.091,810
.125	1.040,194	.264,947	.089,326,5
.1	1.026,02	.259,625	.087,226,6
.075	1.014,753	.255,392	.085,560,4
.05	1.006,62	.252,417	.084,331

$\frac{f}{l}$	D	G
.2	1.350,221,5	1.213,333
.175	1.262,028	1.163,333
.15	1.189,140,6	1.12
.125	1.129,520,5	1.083,333
.1	1.081,845	1.053,333
.075	1.045,555	1.03
.05	1.020,113	1.013,333

Diagrams drawn at an exaggerated scale will give continuous curves from which values of these quantities for any ratio of $\frac{f}{l}$ may be scaled within the limits of the numerical computations. It is clear, however, from the numerical values given that simple interpolation will be sufficiently accurate for all ordinary purposes.

In substituting for the values of the integrals shown in Eq. 41 it will be assumed that the span l_1 is so near $\frac{l}{2}$ as to make it permissible to use that relation, especially as this is usually the case in actual construction. This will make the work expended (either for ordinary or thermal load), in stretching the cable in span l_1, one-half of the work correspondingly expended in span l. A similar observation applies to the work expended in stretching the suspenders in spans l_1 and l. In this manner the resulting equation will be shortened and simplified.

By the use of Eqs. 45 to 57, Eq. 41 may be put in the following form, in which the cross-section of the cable F is considered constant, as for a wire cable.

$$(H'' + \tfrac{1}{2}H')\left\{\frac{C'}{c}(e^{cl} - 1) + \frac{C''}{c}(1 - e^{-cl}) + 2.\frac{C_1'}{c_1}(e^{c_1 l_1} - 1)\right.$$
$$+ 2.\frac{C''_1}{c_1}(1 - e^{-c_1 l_1}) + \frac{1}{H'}\left(\int_0^l M_b\,dx + 2.\int_0^{l_1} M_b\,dx\right)$$
$$+ \frac{t}{E}\left(\frac{3}{2}f + 2v\right)l + \left(1 + \frac{t}{E}\right)\left(\frac{8f}{H'l}(EI + EI_1) - \frac{5}{6}f\,l\right)$$
$$\left. - \frac{H'l^2}{8(f \pm u_c)E}\frac{2lD}{F} \mp \frac{T\delta l^2}{8(f \pm u_c)}2lG\right\} = \frac{l^2}{8(f \pm u_c)}\left\{2\frac{g_0 t}{E}(vA\right.$$
$$\left. + 4fC) + \left(s_0 + \frac{s'}{2}\right)\left(\frac{t}{E} \pm T\delta\right)2\left(v + \frac{f}{3}\right)\right\} \quad . \quad . \quad 58.$$

In this equation $c = \left(\frac{H'}{EI}\right)^{\frac{1}{2}}$ and $C_1 = \left(\frac{H'}{EI_1}\right)^{\frac{1}{2}}$, also $s' = \frac{8H'(f \pm u_c)}{l^2}$.

Or, if X represents the quantity within the brackets in the first member of Eq. 58, *i.e.*, the quantity which is multiplied by $(H'' + \frac{1}{2}H')$, and if Y represents the second member of that equation:

$$H' = 2\left(\frac{Y}{X} - H''\right) \quad . \quad . \quad . \quad . \quad . \quad 59.$$

These equations are perfectly general. They include the effects of moving load on any parts of the three spans separately or concurrently. They are so written as to show the double terms for the two side spans in order that the terms for each such span may be computed separately if desired.

If the stiffening-trusses are non-continuous, and if the moment of inertia I or I_1 is uniform throughout the span, the constants of integration C' and C'' will be given by Eqs. 15*d* and 15*e*. Furthermore, $\frac{t}{E}$ is so small that it may be omitted in the term $1 + \frac{t}{E}$. Then:

$$\frac{C'}{c}(e^{cl} - 1) + \frac{C''}{c}(1 - e^{-cl}) = \frac{16f}{l^2}\left(\frac{EI}{H'}\right)^{\frac{3}{2}}\frac{1 - e^{cl}}{1 + e^{cl}} \quad . \quad 59a.$$

$$2\frac{C'_1}{c_1}(e^{c_1 l_1} - 1) + 2\frac{C''_1}{c_1}(1 - e^{-c_1 l_1}) = \frac{32f_1}{l_1^2}\left(\frac{EI_1}{H'}\right)^{\frac{3}{2}}\frac{1 - e^{c_1 l_1}}{1 + e^{c_1 l_1}} \quad 59b.$$

These values will much simplify the use of Eq. 58.

If it should be considered unnecessary to recognize the stretch of the suspenders, every term in which $\frac{t}{E}$ or $\frac{t}{E} \pm T\delta$ appears is to be made zero. As, however, all those terms, with two exceptions, are simple and independent of H', their presence adds little or nothing to the difficulty of application of Eq. 59.

Inasmuch as $c = \sqrt{\frac{H'}{EI}}$ and as it enters a number of terms in Eq. 58 as an exponent, the value of H' can be determined

through the use of that equation only by trial. That process, however, is not so formidable as it may at first appear. The value of H'', the horizontal component of the cable stress due to dead load, is readily computed as that load is known or can be estimated. The position and amount of the moving load will also be known in each case from which P or w_1 will be determined. The permitted intensity of moving load tensile stress, t, in the suspenders will be specified or it can be determined approximately, while g_0, the weight per linear foot of the cable, can be closely assigned.

The moment of inertia I of the cross-section of the truss, varying if necessary at different points of the span, must first be determined by some approximate method and then corrected, if required, by trial in connection with H'. The quantities $\int_0^l M_b\,dx$ and $2.\int_0^{l_1} M_b\,dx$ become at once determinable when the amount and location of the moving load are known, as M_b is the known moving load bending moment in any span of the stiffening-truss considered as an ordinary beam or truss (continuous or non-continuous, as the case may be), resting on the supports at the extremities of the main and side spans and unconnected with the suspenders. Values for those integrals are given in Art. 75. The remaining quantities involve simply the dimensions of span or structure l, l_1, f, f_1, v, and w_0 the dead load per linear foot below the cable.

After the known numerical computations are thus made and inserted in Eq. 58, the resulting trial equation for the determination of H' will be comparatively simple.

The first approximate value of H' may easily be found by means of the simple procedures of Chap. I, where the theory of the stiffening-truss is treated as a problem in statics. Closely approximate values of the deflection u_c may also be computed by the simple methods of the same chapter.

This approximate value of H' should be decreased to some extent before being inserted in the exponential terms of Eq. 58. The same value of H', if desired, may also be inserted in other terms of X, Eq. 59. The latter equation will then at once

give a value of H' which may be compared with the assumed value of that quantity. This comparison will indicate a trial correction, so as to give a new value of H' to be treated in the same manner as before. This operation can be repeated until the assumed value and that given by Eq. 59 agree sufficiently.

Or again, the assumed approximate value of H' may be substituted in the exponential terms only of Eq. 59 and in the coefficients by which those exponential quantities are multiplied, thus conditioning Eq. 58, so that H' in the remaining terms may be found as from an equation free of exponential terms. The resulting value of H' will indicate whether further trials must be made to secure a sufficient degree of accuracy.

If, as will frequently, or usually, be the case, it is desired to find the horizontal stress H' independently of dead load or temperature, it will only be necessary to make H'', s_0, g_0, and T each equal to zero, and then use the resulting equation.

Again, if in addition to the conditions indicated in the preceding paragraph, the stretching of the suspenders be ignored, Eq. 58 will take the following form, after making $I_1 = r\,I$ and taking the value of $\int M_b\,dx$ from Art. 75:

$$\frac{8f}{l^2}\frac{(E\,I)^{\frac{1}{2}}}{\sqrt{H'}}\left(\frac{1-e^{cl}}{1+e^{cl}}+2\,(r)\,\frac{1-e^{c_1 l_1}}{1+e^{c_1 l_1}}\right)-\frac{H'^2\,l^2}{8(f\pm u_c)\,E}\,\frac{D}{F}-H'\frac{5}{12}\,f\,l$$

$$=-\frac{4f}{l}\,E\,I\,(1+r)-\tfrac{1}{4}\,(l\,\Sigma\,P\,z\,(1-q)+2.\,l_1\,\Sigma\,P\,z\,(1-q)\quad .\quad 59c.$$

This equation may be used for finding the first value of H' for a subsequent and more accurate determination by the use of Eq. 58, if such refinement should be necessary.

Inasmuch as the horizontal stress H'' is due to dead load alone, it can be simply computed without reference to the analysis of this chapter, since all dead load is assumed to be uniformly distributed and to hang directly from the cable at mean temperature. Generally it will be preferable to use Eq. 59*c* instead of Eq. 58, involving H'' and H'. The terms involving the stretch of suspenders may be inserted in Eq. 59*c* without adding any material labor or complication.

It will frequently result in applying Eq. 59*c* that the first

term of the first member and the first term of the second member are so much larger than the others that the first value of H' may be found by equating these two terms only. This approximate value of H' may frequently vary but little from the final value.

This analysis has been written for a single load P placed at any point on the stiffening-truss, which may represent a single panel load or two or three such uniform panel loads if the span is long. The horizontal component H' of the cable stress may then be found for each of all the positions throughout the span and thus represent completely the effect of a uniform moving load covering any part or all of any span. Any further special analysis for uniform loading is, therefore, unnecessary.

The quantity s' in the second member of Eq. 58 is given by Eq. 16*b* or 16*b'* of Art. 66, in which the approximate value of H' is first to be used.

If Eq. 58 is applied for conditions of stress at the mean temperature, $T = 0$, and the last term of the first member will disappear. The plus sign of u_c is to be used for temperatures above the mean and $-u_c$ for temperatures below the mean.

When Eq. 58 is used for temperature stresses only, with the bridge free from moving load, the quantities $\int_0^l M_b\,dx$ and $\int_0^{l_1} M_b\,dx$ depend only on the uniform thermal load. The intensity of tensile stress t in the suspenders will then be due only to the change of temperature.

Art. 75.—Values of $\int_0^l M_b\,dx$ and $\int_0^{l_1} M_b\,dx$.

Values of these quantities both for continuous and non-continuous trusses have been found for all possible loadings in Art. 44, in connection with other quantities required in the formulæ established by the method of Least Work.

If M_2 and M_3 represent the bending moments over the two points of support at the ends of the main or centre span, values of which are given by Eqs. 14 and 15 of Art. 42, Eq. 57 of Art. 44 shows that if the stiffening-trusses are continuous the following values may be written:

For span l_1:

$$\int_0^{l_1} M_b \, dx = \frac{l_1}{2} \left(M_2 + \overset{1}{\Sigma} P z \, (1 - q)\right) \quad . \quad . \quad . \quad 60.$$

For span l_2:

$$\int_0^{l} M_b \, dx = \frac{l_2}{2} \left(M_2 + M_3 + \overset{2}{\Sigma} P z \, (1 - q)\right) \quad . \quad 61.$$

For span l_3:

$$\int_0^{l_3} M \, dx = \frac{l_3}{2} \left(M_3 + \overset{3}{\Sigma} P z \, (1 - q)\right) \quad . \quad . \quad . \quad 62.$$

If the trusses are non-continuous it is only necessary to make the quantities M_2 and M_3 each equal to zero in the preceding equations.

These simple expressions supply all the formulæ needed for the integrals desired.

In accordance with the usual procedure the moment of inertia of cross-section of the truss is assumed to be uniform throughout each span although having different values for different spans. In the ordinary case when the side spans l_1 and l_3 are equal to each other the moment of inertia of the cross-section of truss is assumed to be the same in these two spans, but different from the value of the main or central span l_2.

If the moment of inertia of cross-section of any truss varies materially from one part of the span to another it is sufficiently accurate to determine a mean value to be used for the entire span. While it is feasible to determine all quantities in Eq. 58 affected by the moment of inertia I, even if that quantity be assumed to be variable, such a procedure involves complication and added labor without materially enhancing the accuracy of the results.

If a uniform load w_1 per linear unit (foot) covers a part $x - x_1$ of any span, x and x_1 being measured from the left end of the span and x_1 being less than x, Eqs. 60, 61, and 62 are readily adapted to that condition by writing $w_1 \, dz$ for P in the last term of the second member of each of those equations and then making the proper integration. The result will be:

$$\left.\begin{aligned}\frac{l}{2}\Sigma P z\,(1-q) &= \frac{w_1}{2}\int_{x_1}^{x} z\,(l-z)\,dz = \\ &= \frac{w_1}{2}\left(l\,\frac{x^2-x^2_1}{2}-\frac{x^3-x^3_1}{3}\right) \\ &= \frac{w_1 l^3}{2}\left\{\tfrac{1}{2}\left(\frac{x^2}{l^2}-\frac{x_1^2}{l^2}\right)-\tfrac{1}{3}\left(\frac{x^3}{l^3}-\frac{x^3_1}{l^3}\right)\right\}\end{aligned}\right\}. \quad 63.$$

Eqs. 63 are applicable to any span by simply giving the proper subscript to l.

In the case of non-continuous trusses Eqs. 63 give the value of $\int M_b\,dx$ for a uniform load over any part $x - x_1$ of the span.

If a uniform load covers the entire span, as a thermal load, $x = l$ and $x_1 = 0$. Hence:

$$\frac{l}{2}\Sigma P z\,(1-q) = \frac{w_1 l^3}{12} \quad . \quad . \quad . \quad . \quad . \quad 64.$$

Art. 76.—Value of H for Stiffening-Trusses in Centre or Main Span Only.

This is the frequent case of a stiffened suspension bridge and it is shown in Fig. 2. The stiffening-trusses and suspenders are found in the main span only. The horizontal component H' of cable stress due to moving load is to be found by trial after introducing values of the integrals already determined in Eqs. 45, 46, 48, 49, and 50 of Art. 74. That operation gives:

$$\begin{aligned}&(H''+\tfrac{1}{2}H')\left\{\frac{C'}{c}(e^{cl}-1)+\frac{C''}{c}(1-e^{-cl})+\frac{1}{H'}\int_0^l M_b\,dx\right. \\ &+\frac{lt}{E}(f+v)+\left(1+\frac{t}{E}\right)\left(\frac{8f}{c^2 l}-\tfrac{2}{3}fl\right)-\frac{H'l^3 D}{8(f\pm u_c)EF} \\ &\left.\mp\frac{T\delta l^3}{8(f\pm u_c)}\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)\right\}=\frac{l^3}{8(f\pm u_c)}\left\{\frac{g_0 t}{E}\left(vA+4fC\right)\right. \\ &\left.+(s_0+\tfrac{1}{2}s')\left(\frac{t}{E}\pm T\delta\right)\left(v+\frac{f}{3}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad 65.\end{aligned}$$

Or: $$(H''+\tfrac{1}{2}H')\,X = Y.$$

Hence: $$H' = 2\left(\frac{Y}{X}-H''\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad 66.$$

Every quantity entering the preceding equation may be considered known except H' which, as it enters the exponent c, must be determined by trial. Preliminary approximate values of H' may be estimated by the simple methods of the statically determinate treatment.

If the value of H' is to be found for the mean temperature, $T = 0$.

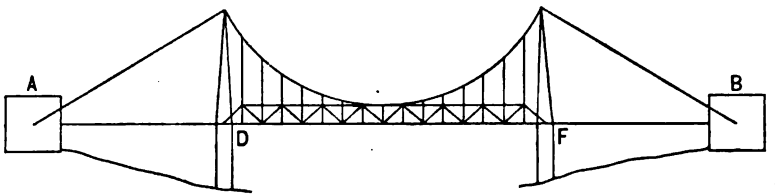


FIG. 2

The ends of the stiffening-trusses at D and F should be firmly anchored to the piers so as to prevent them from leaving their supports vertically, at the same time affording freedom for the horizontal motion of expansion and contraction. The reaction under the ends of the stiffening-trusses at D and F may be either positive or negative, and the greatest possible value of each must be found.

Art. 77.—Application of Analysis to an Actual Bridge Structure.

For reasons stated in the Preface the application of the formulæ of the preceding article to an actual bridge structure will be limited to two illustrations which, however, will be sufficient for the purpose. In this procedure Eq. 59*c* of Art. 74 will be employed and the data to be used belong to the Manhattan Bridge at New York City, already fully set forth for a similar purpose in Art. 59. It will only be necessary, therefore, to repeat here the following quantities: $l_2 = 1446.7$ ft.; $f_2 = 145.3$ ft.; $\frac{f_2}{l_2} = .1$. In this computation the span l_1 has been taken at 720 feet.

It will be convenient in this numerical computation to take

the foot as the fundamental unit, even for the areas of cross-section of the cable and the moments of inertia of cross-section of the trusses; hence $F_0 = 275$ sq. ins. $= 1.91$ sq. ft.

$$I_2 = \frac{44000}{144} = 305.6; \quad I_1 = \frac{51000}{144} = 354.2.$$

Hence: $$r = I_1 \div I_2 = 1.16.$$

In the tabulation following Eq. 215 in Art. 59, there is given the horizontal component H in the non-continuous main span of the Manhattan Bridge produced by a unit moving load placed at the various points in the span indicated. In this case there will be found the horizontal component H' in the same span found by placing a panel load of 10,000 lbs. at the one quarter and centre points of the span. These two values of H' will therefore be, by the tabulation of Art. 59:

For $q = \frac{1}{4}$ (quarter point):

$$H = 10{,}000 \times 1.276 = 12{,}760 \text{ lbs.}$$

For $q = \frac{1}{2}$ (centre point):

$$H = 10{,}000 \times 1.791 = 17{,}910 \text{ lbs.}$$

Trial values of H' are to be placed in the first member of Eq. 59c, Art. 74, while in the second member the summation sign in the last term of that member is to be omitted. The second term of the second member will then become:

$$-\frac{l_2}{4} P z (1 - q) = -\frac{l_2^2}{4} P q (1 - q).$$

The l_2 in this equation has been taken as 1447 ft.

As the foot is the unit, the modulus of elasticity $E = 30{,}000{,}000 \times 144 = 4{,}320{,}000{,}000$. Hence:

$$EI = 1{,}319{,}904{,}000{,}000.$$

The two quantities c and c_1 have the values:

$$c = \frac{H'}{EI}; \quad c_1 = \frac{H'}{EI_1}.$$

With the panel load of 10,000 placed first at the quarter

point and then at the centre point of the span it is necessary to find two values of H' which will satisfy Eq. 59*c* with the second member modified as explained above. The two values of H' found by the Least Work Method and shown in the tabulation following Eq. 215 of Art. 59 may serve as guides in the tentative efforts to reach the values desired, since Eq. 59*c* is exponential in relation to the horizontal cable stress H'. The use of logarithms will simplify and expedite this tentative operation, which at best, however, is laborious.

The panel load $P = 10{,}000$ lbs. will first be taken at the quarter point, making $q = \frac{1}{4}$. The assumption of $H' = 12{,}200$ lbs. will be made. Substituting this value and the other quantities required, and as given above, in Eq. 59*c* of Art. 74, it will be found that the large parenthesis containing the exponential quantities in the first term of the first member of that equation will be .14977. Then substituting the numerical values of EI, f_2, l_2, and taking $u_c = 0$, the entire first member of the equation will take the value:

$$- 1{,}138{,}066{,}600{,}000.$$

The second term of the first member of Eq. 59*c* will be found to have such a small value in comparison with the first term that it may be omitted without appreciable error. In fact, the first term, $- H' \frac{5}{12} f_2 l_2$, might also be omitted without material error as it would have to be multiplied by more than 1,000 in order to be as large as the first term of the first member of Eq. 59*c*.

Again, completing the numerical substitutions in the second member of the same equation, it will be found that it takes the value:

$$- 1{,}141{,}381{,}000{,}000.$$

These two numerical values of the first and second members of the equation under consideration differ by not over $\frac{3}{10}$ of 1% and they may be considered sufficiently near each other to

give the correct value of H', *i.e.*, 12,200 lbs. Trial was made with two other values only of this quantity, but more would have been necessary if that established by the Least Work Method were not available.

It was found that the second term of the second member of Eq. 59*c* in the preceding numerical operation was less than the one-thousandth part of the first term, and hence that it might be omitted without material error. This last result shows that the second member of Eq. 59*c* varies but little with varying positions of the single panel load.

Again, taking the panel P = 10,000 lbs. at the centre of the main span l for which, by the Least Work Method, H' = 17,910 lbs., let the trial value H' = 16,500 lbs. be assumed. After making the numerical substitutions precisely as in the previous case, making, however, $q = \frac{1}{2}$, the first member of the equation will take the value:

$$- 1{,}137{,}742{,}700{,}000.$$

The numerical substitution of the proper quantities, in the second member of the equation, will give the result:

$$- 1{,}141{,}708{,}600{,}000.$$

The difference between these two results is about $\frac{3}{10}$ of 1% and the corresponding value H' = 16,500 lbs. may be considered satisfactory. Two trials only were required to reach this solution.

These two illustrations exhibit the method of application of either Eq. 59*c* for the determination of the horizontal moving load cable stress H' or the more nearly exact Eq. 58. The results of the two methods differ between 4% and 5% with the panel load at the quarter point and about 8% when the panel load is at the centre point of the span, the Least Work Method giving the largest value in both cases. As Eq. 59*c* is an approximate equation, this comparison of results may be considered satisfactory. In fact, in view of the conditions existing in an actual large suspension bridge structure, the results obtained by either method may be accepted as essentially correct.

CHAPTER VI.

THERMAL STRESSES

Art. 78.—Temperature Stresses in Stiffened Suspension Bridges.

Centre Sag of Cable, Equivalent Horizontal Component of Cable Stress and Suspender Pull.

A variation in temperature of the cables and suspenders of a suspension bridge will change their lengths, increasing or decreasing the sag of the cable and thus causing a corresponding deflection downward or upward of the stiffening-trusses. The resulting effects may be supposed to be the same as those produced by that uniform load over the entire span or spans which would cause the same variation of length of the cables as the change in temperature, but it must be remembered that the thermal change of cable length displaces that ordinarily produced by the cable stress, the actual thermal suspender pull corresponding to that part of the ordinary loading carried by the stiffening-truss as an ordinary beam.

At this point it is essential to observe the difference between the two cases of a cable with a uniform section, like a wire cable, and one with a section varying in proportion to the resultant cable stress, like an eye-bar cable. In the former case the intensity of stress in any normal section of the cable varies in direct proportion to the total or resultant stress in the section, but in the latter case the intensity is constant. The rate of change of length of any part or all of the cable due to change of temperature is uniform for a given range of temperature irrespective of the cross-section. This is identical with the change in length of cable due to the stress in it only when the intensity of cable stress is uniform in all sections. This latter condition does not exist in the cable with uniform section. The uniform load, p, there-

fore, corresponding to a given thermal centre deflection, will not be quite the same for the two cases.

If it be supposed that the towers of the main span are of equal height, then as heretofore let L be the length of the main cable at mean temperature between the tops of the towers. If T be the variation of the temperature (Fahrenheit) from the mean, and if δ be the coefficient of thermal expansion or contraction for the Fahrenheit scale, *i.e.*, the change in length for each linear unit (foot) per unit of temperature, the change in length for T degrees change in temperature will be:

$$\Delta L = T \delta L \quad . \quad . \quad . \quad . \quad . \quad 67.$$

The corresponding variation in elevation of the vertex (lowest point) of the cable, given by Eq. 82 of Art. 35, will be:

$$\Delta f = \frac{15}{16(5n - 24n^3)} \Delta L \quad . \quad . \quad . \quad . \quad 68.$$

In this equation $n = \frac{f}{l}$.

If the saddles at the tops of the towers move toward the centre of the span so as to shorten the latter by the amount Δl, the centre sag of the cable will be increased to the extent shown by Eq. 89 of Art. 35:

$$\Delta' f = \frac{15 - 8(5n^2 - 36n^4)}{16(5n - 24n^3)} \Delta_{\prime} \quad . \quad . \quad . \quad 69.$$

When the tops of the towers move horizontally in consequence of the flexure of the latter, the centre sag of the cable will be affected precisely as shown by the value of $\Delta' f$.

If the section of the cable varies as the resultant cable stress, F_0 being the sectional area at the lowest point (vertex) of the cable where H' (the horizontal component of the resultant stress throughout the cable producing the same total stretch as the change of temperature) exists, the rate of stretch of cable due to the stress in it is constant throughout its length. Hence:

$$\frac{H'}{F_0 E} L = T \delta L \therefore H' = F_0 T E \delta \quad . \quad . \quad . \quad 70.$$

If the section of the cable is constant:

$$\frac{H'}{F_0 E}\int_0^l \left(\frac{dy^2}{dx^2}+1\right)^{\frac{1}{2}}\left(\frac{dy^2}{dx^2}+1\right)^{\frac{1}{2}}dx = \frac{H'}{F_0 E}l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right) = T\delta L \,. \quad 71.$$

$$\therefore H' = F_0 T \frac{E\delta}{1+\frac{16}{3}\frac{f^2}{l^2}}\frac{L}{l} \quad . \quad . \quad . \quad . \quad . \quad 72.$$

If there be taken for steel wire cables, $\delta = .000,006,2$ and $E = 30,000,000$ lbs. per square inch, the following tabular results will be found:

$\frac{f}{l}$	$\frac{E\delta}{1+\frac{16}{3}\frac{f^2}{l^2}}\frac{L}{l}$	$\frac{f}{l}$	$\frac{E\delta}{1+\frac{16}{3}\frac{f^2}{l^2}}\frac{L}{l}$
.2	168.347,47	.1	180.176,97
.175	172.116,63	.075	183.245,89
.15	175.520,86	.05	184.767,72
.125	178.594,33		

In these computations the value of $\frac{L}{l}$ is taken from the tabulated statement of Art. 5, based upon Eq. 17*a* of that article.

Having determined the value of what may be called the equivalent thermal horizontal cable stress H' by one of the preceding equations, the corresponding suspender pull s' per linear unit (foot) of span follows at once if the centre sag or depression of the cable at the temperature T be represented by f':

$$s' = \frac{8f'}{l^2}H' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 73.$$

The quantity f' will evidently have the value:

$$f' = f + \Delta f + \Delta' f \quad . \quad . \quad . \quad . \quad . \quad 74.$$

The proper signs of Δf and $\Delta' f$ must be carefully observed. The term $\Delta' f$ should include the centre sag due to a movement

of the tops of the towers toward each other in consequence of flexure or other cause, if there is any.

The length of the shortest suspender at the centre of the span is v, and the sectional area of suspenders per unit of length of span is F_2. Hence the stretch of that suspender is:

$$\frac{s'}{F_2 E} v = \Delta_2 f \quad . \quad . \quad . \quad . \quad . \quad 75.$$

Thermal Centre Deflection of the Continuous Stiffening-Truss and Thermal Uniform Load p_t.

There must be found a uniform load p_t which, extending over each entire span, will produce a centre deflection of the stiffening-truss, equal, in general, to $\Delta f + \Delta' f + \Delta_2 f = u_t$. That load is called the uniform thermal load.

Let it be supposed that the spans l_1 and l_3 are equal to each other and that they are entirely covered by the uniform moving load p_t and, further, that $a = b$ and $r = \frac{I}{r'}$, while the uniform load p_t covers also the whole of l_2. Then if these conditions be introduced in Eqs. 86 and 87 of Art. 47:

$$M_3 = p_t \frac{l_2^2}{4} \frac{a^3 + r}{2a + 3r} = M_2 \quad . \quad . \quad . \quad . \quad 76.$$

Representing the centre deflection of the main span l_2 by u_t, then resorting to Eq. 119, Art. 51, in which x must be made $\frac{l_2}{2}$, at the same time taking C from Eq. 120 of the same article, and $M_2 = M_3$ as given above, there will result:

$$E I_2 u_t = (p_t - s') \frac{l_2^4}{32} \frac{a^3 + r}{2a + 3r} + H' \frac{5 f_1 l_2^2}{48} - p_t \frac{5 l_2^4}{384} \quad . \quad 77.$$

Or, since $H' = \frac{s' l_2^2}{8 f_2^1}$ . . . . . 78.

$$E I_2 u_t = (p_t - s') \frac{l_2^4}{32} \left(\frac{a^3 + r}{2a + 3r} - \frac{5}{12} \right) \quad . \quad . \quad 79.$$

As s' is given by Eq. 73, and as:

$$u_t = \Delta f_2 + \Delta' f_2 + \Delta_2 f_2 \quad . \quad . \quad . \quad . \quad . \quad 80.$$

$$\therefore p_t = \frac{E I_2 u_t}{\frac{l_2^4}{32}\left(\frac{a^3 + r}{2a + 3r} - \frac{5}{12}\right)} + s' \quad . \quad . \quad . \quad 81.$$

p_t is thus completely determined.

In these equations, obviously:

$$f_2' = f_2 + \Delta f_2 + \Delta' f_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad 82.$$

Similar equations are clearly applicable to the spans l_1 and l_3 by taking corresponding values for f_1 and f_3, *i.e.*, by measuring the sag from the inclined chord as shown in Fig. 10, Art. 42, and considering the moment zero at the left end of span l_1 and right end of l_3, but they are not needed here.

In the ordinary case $a = \frac{1}{2}$, and for that value:

$$\frac{a^3 + r}{2a + 3r} = \frac{.125 + r}{1 + 3r} \quad . \quad . \quad . \quad . \quad . \quad 83.$$

If the moment of inertia is the same in the three spans, $r = 1$ and the preceding fraction will have the value .28125.

It is interesting to observe that if $s' = 0$, and if continuity disappears, or making $\frac{a^3 + r}{2a + 3r} = 0$:

$$u_t = -\frac{5 p_t l_2^4}{384 E I_2}.$$

This is the proper deflection of a simple beam supported at each end and loaded uniformly with p_t per linear unit.

Having thus determined p_t, s' and the horizontal stress H', the bending moments, shears and all stresses in the stiffening-trusses, towers or other parts of the structure due to any variation of temperature whatever, may at once be computed by the aid of the preceding equations of this chapter. It is only necessary to observe that the temperature may vary in either direction from the mean and that the bending moments and shears will be of opposite signs for upward and downward deflections, although

the equations of this article are applicable for deflection in either direction.

Thermal bending moments, shears, and other quantities are to be computed independently of the effects of vertical loading, after which the results so obtained may readily be combined with those due to dead and moving or other loads.

Thermal Centre Deflection of Non-Continuous Stiffening-Truss and Thermal Uniform Load p_t.

The equations for this case follow from those of the preceding case in the simplest possible manner.

Eq. 72 or 78 gives the horizontal component H' of the cable stress, while the centre deflection u_t of the stiffening-truss has precisely the value given by Eq. 88. The suspender pull per linear unit (foot) of span has the same value already given:

$$s' = \frac{8 f'_2 H'}{l_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 84.$$

f'_2 is given by Eq. 82.

As the moments M_2 and M_3 at the ends of the span are now zero, Eq. 77 becomes:

$$E I_2 u_t = -(p_t - s') \frac{5 l_2^4}{384} \quad . \quad . \quad . \quad . \quad 85.$$

$$\therefore p_t = -\frac{E I u_t}{\frac{5 l_2^4}{384}} + s' \quad . \quad . \quad . \quad . \quad 86.$$

The preceding formulæ may be used as they stand for the spans l_1 and l_3, remembering that f_1 and f_3 must be measured downward from the inclined chords or diameters drawn from the tops of the towers to the extremities of those spans.

It is important to observe in the preceding equations that $p_t - s'$ is the true thermal suspender pull per linear unit of span, *i.e.*, it is that part of the uniform load p_t which is carried by the trusses and produces their deflection. Thermal variation does not directly affect cable stress due to that part of the load carried direct by the cable, as the latter is perfectly flexible.

Art. 79.—Temperature Stresses Determined by the General Value of the Horizontal Component H of the Cable Stress.

In the preceding article the horizontal component H' of cable stress has been determined on the assumption that a moving uniform load over each entire span rests on the stiffening-trusses so as to produce the same change of length of the cable as the assumed change of temperature. As this leads to the same deflection of stiffening-trusses as the corresponding thermal change of cable length, the bending moments, shears, and deflections of those trusses are the same as the true thermal effects, but the cable and suspender stresses are much too large and they must be found by other methods. If all moving and dead load be considered zero in Eq. 36, Art. 43, and if suitable terms expressing thermal effects be added to the tabulated statement preceding Eq. 36, the resulting value of H due to a change of temperature will be independent of all external loading whatever, and that value will now be found.

It will be noticed, if M_b, k, and w_0 have each the value zero, that the value of H becomes zero also, as it should. It is necessary to add to the quantities in the tabulated statement under the head of $E\frac{dX}{dH}$ similar quantities due to the change of length of the cable and suspenders, as well as of the two towers, caused by the change of temperature of T degrees. If δ represents the coefficient of thermal contraction or expansion of steel in the cables, suspenders, and towers, the change of length of the element dL of the cable corresponding to the assumed change of temperature will be $T\,\delta\,dL$. If H' is the horizontal component of cable stress sought, then the resultant tension R in the cable at any point will be $R = H'\frac{dL}{dx}$.

Referring back to the first term of the second member of Eq. 29, Art. 43, there may be written for the cable of the main span:

$$R = H'\frac{dL}{dx};\quad \frac{R}{EF} = T\,\delta, \text{ and } \frac{dR}{dH'} = \frac{dL}{dx}.$$

Hence,

$$\int \frac{R}{F}\frac{d\,R}{d\,H}d\,L = \int^{l} E\,T\,\delta\,\frac{d\,L^2}{d\,x^2}\,d\,x = E\,T\,\delta\,l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)$$

For the purposes of this investigation it will be assumed that each of the two side spans may be taken half the length of the main span without material error, and hence that the thermal work expended in changing the length of the two side span cables is equal to that expended on the main span cable. Again, it will be assumed that each side span cable curves down to the roadway floor just before it enters its anchorage, *i.e.*, at the span length l_1 from the tower, in which case $f + v = h''$. Remembering these observations and following the procedure illustrated above for the main span cable, there will result the set of expressions found below which must in general be added to the last column of the tabulated statement from which Eq. 36, Art. 43, is written:

Part of bridge,	$E\,\dfrac{d\,X}{d\,H'}$
Cable in main span	$E\,T\,\delta\,l\left(1+\dfrac{16}{3}\dfrac{f^2}{l^2}\right)$
Two cables in side spans	$E\,T\,\delta\,l\left(1+\dfrac{16}{3}\dfrac{f^2}{l^2}\right)$
Suspenders in main span	$E\,T\,\delta\,l\dfrac{8\,f}{l^2}\left(v+\dfrac{f}{3}\right)$
Suspenders in two side spans	$E\,T\,\delta\,l\dfrac{8\,f}{l^2}\dfrac{h''}{3}$
Two towers	$2\,E\,T\,\delta\,l\dfrac{h'}{l}\,(\tan\alpha+\tan\alpha_1).$

If the side spans are simply backstay spans of length l_1:

$$E\,\frac{d\,X}{d\,H'} = 2\,E\,T\,\delta\,l_1\sec^2\varphi,$$

and this expression must displace the quantity $E\,T\,\delta\,l\left(1+\dfrac{16}{3}\dfrac{f^2}{l^2}\right)$ given above for the two cables in the side spans.

At the same time the quantity given for "Suspenders in two side spans" must be omitted.

For this condition of thermal stresses M_b and w_o must each be zero in the values of the moment M for the stiffening-trusses in both the main and side spans in the tabulated statement from which Eq. 36 is derived, while M'' and M' must be taken from Eqs. 19 and 20, in which $s' = H' \frac{8f}{l^2} = H' \frac{8f_1}{l_1^2}$. After placing the sum of all the terms of the kind $E \frac{dX}{dH'}$, thus determined, equal to zero, the following value of thermal H will be found for continuous trusses and suspenders in all three spans:

$$H' = -\frac{E I T \delta l \left\{ 2\left(1 + \frac{16}{3}\frac{f^2}{l^2}\right) + \frac{16 f}{3 l^2}(2v + f) + \frac{2 l_1}{l}(\tan\alpha + \tan\alpha_1)\right\}}{\frac{8}{15}\left(f^2 l + \frac{2}{r} f_1^2 l_1\right) + \frac{16}{3}\frac{f^2}{l}(B - C) + \frac{64 f^2}{3 l^3}\left(B^2 - C(B-C)\right) - \frac{32}{3 r}\frac{f_1^2}{l_1} C\left(1 - \frac{4C}{l_1^2}\right)} \quad 87.$$

If the stiffening-trusses are non-continuous, the numerator of the preceding equation (represented below by $E I T \delta l N$) will remain unchanged, but in the denominator $B = C = 0$.

Hence for non-continuous trusses:

$$H' = -\frac{E I T \delta l N}{\frac{8}{5}\left(f^2 l + \frac{2}{r} f_1^2 l_1\right)} \quad . \quad . \quad . \quad . \quad . \quad 88.$$

If T represents an increase in temperature it will be positive and H' will be negative, thus showing a decrease in cable stress and indicating that the cable will be relieved of a corresponding amount of dead or moving load, or both, which, in consequence of the increased sag, must be taken as increased load by the stiffening-trusses acting as ordinary trusses continuous or non-continuous.

On the other hand, if T is negative, H' will be positive, indicating an increase in cable stress by taking up a corresponding part of the moving load carried by the stiffening-trusses at mean temperature.

These considerations show that an increase of temperature will affect (generally increase) the bending moment M in the stiffening-trusses, even in the absence of moving load, by imposing dead load (or dead and moving load) on those members; and that a decrease of temperature will in general relieve the moment M in the stiffening-trusses by taking up a corresponding amount of moving load carried by those trusses at mean temperature.

After H' is thus found, the true thermal suspender pull s' will be:

$$s' = H' \frac{8f}{l^2} = H' \frac{8f_1}{l_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad 89.$$

With an increased temperature H' and s' are both negative, indicating that the bending effect on the stiffening-trusses is the same as if the suspenders were in compression and exerting a downward thrust on those trusses.

The suspender pull s' per linear unit of span given by Eq. 89 is the same as the quantity $p_t - s'$ of Eqs. 81 and 86 of the preceding article; the latter being that part of the equivalent thermal load carried by the stiffening-trusses as ordinary beams or trusses.

Application to Actual Structure Used in Art. 59.

The coefficient of thermal expansion or contraction per-degree Fahr. will be taken as

$$\delta = .000,006,2$$

The foot will be taken as the linear unit. Hence:

$$E = 144 \times 30,000,000 = 4,320,000,000.$$

The other data required, taken from Art. 59, are:

$l = 1450$ ft.; $f = 145$ ft.; $\frac{f}{l} = .1$

$l_1 = 720$ ft.; $f_1 = 37.2$ ft.; $\frac{f_1}{l_1} = .0517$

$h_1 = 282$ ft.; $h'' = f + v = 175$ ft.

$v = 30$ ft.; $tan\,\alpha + tan\,\alpha_1 = .8617$; $T = \pm 45°$ Fahr.

$I = 305.6$ ft.; $r = \frac{I_1}{I}$; $E\,\delta = 26{,}784.$

$F = 275$ sq. inches $= 1.91$ sq. ft. for one cable.

The stiffening-trusses are in all three spans and they are non-continuous, hence the data given above must be substituted in Eq. 88. The result, if $T = -45°$, is:

$$H' = 99{,}350 \text{ pounds.}$$

The thermal suspender pull s' per linear foot is:

$$s' = H' \frac{8f}{l^2} = H' \frac{8f_1}{l^2_1} = 54.8 \text{ pounds.}$$

These are the thermal values of H and s' for a decrease of temperature of 45° from the mean.

Eq. 72 of the preceding article, with the aid of the tabular statement following that equation, shows that H' due to the equivalent thermal load p_t is, if $\frac{f}{l} = .1$:

$$H' = 180.177\, F_0\, T = 2{,}229{,}700 \text{ pounds.}$$

The value of s' is, therefore:

$$s' = H' \frac{8f}{l^2} = 1230 \text{ pounds.}$$

In the present instance the sectional area of a single cable is 275 square inches. Hence the equivalent thermal intensity of stress in the cable at the top of the tower, where $sec\,\alpha = 1.077{,}27$, is:

$$\frac{2{,}229.700 \times 1.07727}{275} = 8{,}735 \text{ lbs. per square inch.}$$

The value of H' will be somewhat, though not largely, affected by the decrease or increase of f due to the variation of temperature, as shown in the preceding article, but as that influence is relatively small it has been neglected in these computations.

Value of H' for Uniform Equivalent Thermal Load p_t of Stiffening-Trusses.

It may be desirable in some cases to consider the effect of the equivalent uniform thermal load p_t per linear unit (foot) over each entire span. This may be done by employing Eqs. 36, 69, 70, or 75 of Chap. IV. In such a procedure the uniform load p_t will be considered the moving load producing the bending moment M. As the effects of the uniform load p_t only are desired, w_0 and k must each be made equal to zero, thus causing to disappear from the equations used all terms in which those letters occur.

The bending moments M'' and M' have the values already used. The only remaining quantities to be employed are the terms:

$$-\int_0^l M_b\, y\, dx;\ -\frac{8f}{l^3}\left((B+C)\int_0^l M_b\, x\, dx - C\, l\int_0^l M_b\, dx\right);$$

$$-\frac{I}{r}\int_\infty^{l,l_1} M_b\, y_1\, dx;\ +\frac{8f_1}{r\, l_1^3}\, C\int_\infty^{l,l_1} M_b\, x\, dx.$$

The equations of Art. 44 give these values at once, remembering that $a = b = \frac{1}{2}$ and $r = \frac{I}{r'}$.

If the uniform thermal load p_t covers all three spans, Eqs. 86 and 87 of Art. 44 give:

$$M_3 = M_2 = -\,p_t\frac{l^2}{32}\frac{8r+1}{3r+1} = -\,p_t\frac{l_1^2}{8}\frac{8r+1}{3r+1} \quad . \quad 90.$$

The first of Eqs. 55 of Art. 44 then becomes:

$$-\int_0^l M_b\, y\, dx = -\,p_t\frac{f\, l^3}{240}\frac{8r+11}{3r+1} \quad . \quad . \quad . \quad . \quad 91.$$

Similarly, Eqs. 56 of Art. 44 show:

$$\int_0^l M_b x\,dx = p_t \frac{l^4}{192}\frac{5}{3r+1} \quad . \quad . \quad . \quad . \quad 92.$$

Also, by using Eq. 57:

$$\int_0^l M_b\,dx = p_t \frac{l^3}{96}\frac{5}{3r+1} \quad . \quad . \quad . \quad . \quad 93.$$

$$-\frac{8f}{l^3}\left((B+C)\int_0^l M_b x\,dx - Cl\int_0^l M_b\,dx\right)$$

$$= -p_t \frac{fl}{24}(B-C)\frac{5}{3r+1} \quad . \quad . \quad . \quad . \quad 94.$$

The second of Eqs. 55 gives:

$$-\frac{2}{r}\int_0^l M_b y_1\,dx = p_t \frac{f_1 l_1^3}{60r}\frac{16r-3}{3r+1} \quad . \quad . \quad . \quad 95.$$

Finally, using the first of Eqs. 56:

$$\frac{8f_1}{r l_1^3} C \int_{00}^{l_1 l_1} M_b x\,dx = -p_t \frac{f_1 l_1}{3}\frac{5}{3r+1} C \quad . \quad . \quad 96.$$

If the stiffening-trusses are non-continuous, $B = C = 0$, and Eqs. 91 and 95, only, are required.

Value of Thermal H' Found by the General Formulæ of Art. 74.

The formula, Eq. 58, for the horizontal component of cable stress established by the method of deflections includes the terms expressing the effects of a variation of temperature in the cables and suspenders. Hence the value of thermal H' may be found by making all terms in that equation depending upon the moving or dead load zero, leaving only those terms depending on a change of temperature from the mean. The resulting equation may then be solved for H' which will represent the horizontal component of thermal stress in the cables. Inasmuch, however, as the H' desired is a factor in the exponential terms of the equation under consideration the operation would involve

the usual complicated solution by trial which has already been illustrated for vertical loads. Hence, an illustration of this method of determining H' will not be given. It is difficult to justify the use of such a complicated equation when other simple and equally accurate methods may be employed.

Thermal H' Found by Method of Art. 78.

As $p_t - s'$ used in the preceding article and corresponding to the equivalent horizontal component of cable stress is equal to the uniform suspender pull which produces the true thermal horizontal component H' of cable stress, Eqs. 81 and 86 of that article may be employed to find the true suspender pull producing the thermal H'. It is necessary, however, first to find u_t, the deflection at the centre of the span produced by the variation of temperature.

Precisely the same data as those of the preceding section of this article will again be employed here. Inasmuch as the stiffening-trusses are non-continuous and as the stiffening-trusses and suspenders are found in all three spans, Eq. 86 of the preceding article must be used. A variation of $+ 45°$ Fahr. will be assumed and the effects of that variation on the towers will be recognized, although it is small. Again, as the length of the shortest suspender at the centre of the main span is about 30 feet only, the effect of the thermal variation in its length on the centre deflection of the truss will be ignored.

Making the numerical substitutions of the data in Eq. 67 of Art. 78, the variation in the length of cable in the main span will be:

$$\Delta L = .415 \text{ ft.}$$

The same substitutions in Eq. 68 of that article will give:

$$\Delta f = .8175.$$

A positive thermal variation in length of the cables in the side spans will cause a movement of the saddles at the tops of the towers toward the centre of the main span equal to the sum

of the lengths of the side spans multiplied by $+ T\,\delta$. Inasmuch as the length of each side span is taken to be 720 ft., the main span will be shortened by .40176 ft. $= \Delta\, l$, making this and the other proper numerical substitutions in Eq. 69 of the preceding article, there will result:

$$\Delta' f = 1.92 \text{ ft.}$$

The height of a tower above the point where the stiffening-truss rests upon it is about 175 feet, hence that part of the tower will be lengthened by $175\ T\ \delta = .0488$ ft. As this is a lengthening of the tower the central deflection will be $u_t = .8175 + 1.92 - .0488 = 2.6887$ ft.

Substituting the preceding numerical values, including u_t, in Eq. 86 of Art. 78, there will result:

$$p_t - s' = -\frac{384\,E\,I\,u_t}{5\,l'^4_2} = 61.6 \text{ lbs. per lin. ft.}$$

As the value of this uniform load computed by the Method of Least Work is seen above to be 54.8 lbs., the agreement between the two methods is satisfactory for such a computation.

This method has the advantage of simplicity and gives results of sufficient accuracy for thermal computations.

General Observations.

After thermal H' and s' are found by Eqs. 87, 88, and 89, the thermal moments, shears, and deflections, indeed thermal effects of every kind, become completely known. The formulæ giving these quantities and set forth in the preceding articles of Chapters IV and V are immediately applicable for all thermal computations precisely as for ordinary vertical loads. These thermal effects computed independently of other load effects are then to be combined with the computations for ordinary vertical loads in such manner as to produce the greatest resultants possible.

Stresses Due to Other Changes of Length of Cables.

Any change of length of cable due to an increase of stress or decrease of stress will change the deflection of the stiffening-truss and produce moments, shears, and other stresses corresponding to such changes in length. The resulting moments and stresses due to such variations of cable length belong to the same class as thermal stresses, and they may readily be computed by the formulæ already established for thermal stresses. It is only necessary to substitute for the thermal changes in length of the cable the changes due to the increased or decreased stresses in it. This is exactly equivalent to writing the rate of strain for $T\,\delta$ in the equations for thermal stresses already established. No other change is necessary, and the procedures required to determine the corresponding stresses are precisely the same as those already illustrated for changes of temperature.

CHAPTER VII

Arch Ribs Treated By Graphical Methods

Art. 80.—Equilibrium Polygons, or Polygonal Frames.

When the sides of an equilibrium polygon take the positions and lengths of the centre lines of a constructed frame, the equilibrium polygon is called a polygonal frame for the loads used in drawing that polygon. Preliminary to the specific treatment of arch ribs by the graphical method it is advisable to outline here some elementary general principles regarding equilibrium polygons for any given system of vertical forces or loads, although they will be found fully set forth in any published treatment of the general subject of graphic statics. Similarly, for convenience of reference, some elementary and general matters concerning the common theory of flexure will also be set forth in a later article.

In the figure below, let $A\ L$ be any straight, simple beam, subjected to the action of vertical forces at B, C, D, etc. Let x be measured from any section positive and horizontal toward A, and let P signify any external force such as the reaction at A or any of the loads applied to the beam, then will $\Sigma\ P\ x$ represent the bending moment to which the beam is subjected at the section located by x. Let there be imagined any force T acting parallel to $A\ L$, and let the moments $\Sigma\ P\ x$ be taken at each of the points B, C, D, E, F, G, K. Then, if the quotients of those moments divided by the horizontal component H, of T, be supposed represented by the vertical lines $b\ B$, $c\ C$, $d\ D$, etc., respectively, will the polygon $A\ b\ c\ d\ e\ f\ g\ k\ L$ be *one* equilibrium polygon for the given system of loads; so that if the beam $A\ L$ were displaced by a tie in which exists the stress T, with the

given loads hung from the joints b, c, d, etc., the whole system would be in equilibrium.

In order to establish this, it is only necessary to show that no piece of the polygon is subjected to bending, for if that is

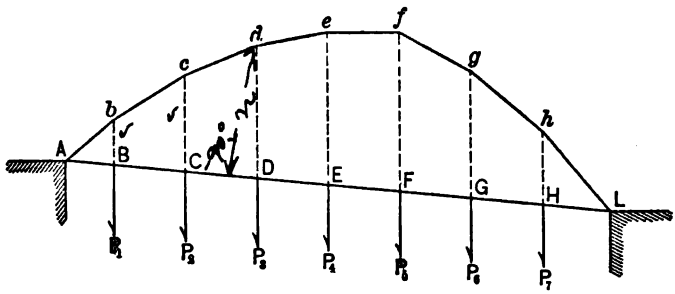


FIG. 1.

the case the line of action of the resultant stress must coincide with its centre line.

Consider any portion of the system, as that lying on the left of the vertical line $d\,D$. Those forces which have moments about the point d are the external forces to the left of $d\,D$ and the stress T in the tie $A\,L$. The latter has a lever-arm n, equal to the normal distance from d to $A\,L$, and its moment is opposite in sign to $\Sigma\,P\,x$. Consequently the resultant moment about d will be $M = \Sigma\,P\,x - T\,n$. But by construction $\Sigma\,P\,x = T\,n$, hence $M = 0$ and the same result is true of every other joint. If H is the horizontal component of T, then evidently $T\,n = H\,(d\,D) = \Sigma\,P\,x$.

If v be the general representative of the vertical ordinates $b\,B$, $c\,C$, etc., then, in general,

$$v = \sum \frac{P\,x}{H} = \frac{1}{H} \sum P x.$$

The last member of this equation is written because H is a constant quantity. From these considerations follows this important principle:

The vertical ordinates of the equilibrium polygon of any system of vertical loads are proportional to, and may represent,

the bending moments found at the various sections of a straight beam subjected to the action of the same system of loads, and having the same span.

Since the stress T was taken arbitrarily, it is evident that there may be an indefinite number of equilibrium polygons for any given system of loads, each having a value of T or H differing from the others; but the principle stated above is perfectly general, and is true for all.

The equation above shows that v varies inversely as H.

Art. 81.—Construction of Equilibrium Polygon or Polygonal Frame for Vertical Loads.

Equilibrium Polygon for Frame or Rib with Hinged Ends.

The method of constructing the equilibrium polygon given in the preceding article is not the most convenient, nor the one commonly used. The ordinary method is the following:

Let $A\,K$, Fig. 2, represent the centre line of a beam simply supported at A and K and inclined in this case, but ordinarily horizontal; also let P_1, P_2, P_3, etc., be the vertical loads acting

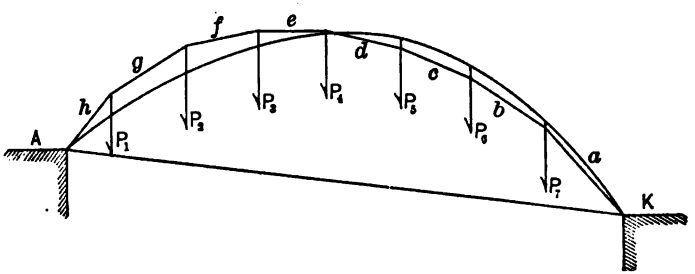


FIG. 2.

along their respective lines of action. In Fig. 3 let the parts 1, 2, 3, 4, 5, 6, and 7 of the vertical line 1–7 represent those loads taken by any assumed scale. Since $B\,C$ represents the sum of all the applied loads, it is also equal to the sum of the two reactions or end shearing stresses at A and K. In the case of the

simple beam taken those quantities will be determined by the law of the lever only.

Suppose $A^1 C$ and $A^1 B$ to represent the end shears or upward reactions at A and K, respectively. Then draw $A^1 P$ parallel to $A K$, and on it take any point P. From P draw the radial lines $a, b, c, d, \ldots h$, as shown, and starting from A or K in Fig. 2, draw the lines $a, b, c, d, \ldots h$ parallel to the lines denoted by the same letters in Fig. 3. Then will Fig. 2 represent an equilibrium polygon for the given span and system of loading.

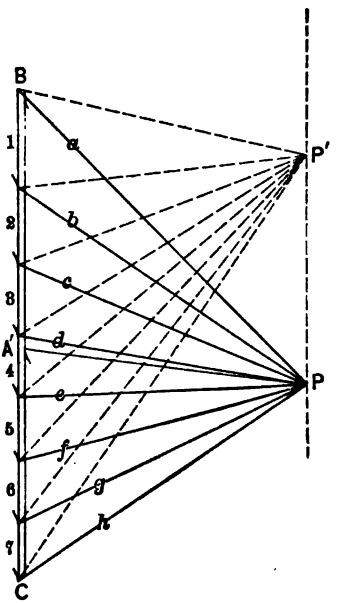


FIG. 3.

The line $A K$ or $P A^1$ is called the *closing line* of the polygon. The reaction at A is evidently composed of the numerical sum of the vertical components of stress in h and $A K$, while that at K is equal to the numerical difference of the vertical components of stress in a and $A K$.

The point P, from which the radial lines are drawn, is called the *pole*, and the normal distance from the pole to the load line $B C$, the *pole distance*. The pole distance evidently represents the horizontal component of stress common to all the members of the polygon.

In order that the equilibrium polygon, constructed according to the principles given above, shall exactly fit the span, it is only necessary that a proper observance be paid to the scales used.

The equation $v = \frac{\Sigma P x}{H}$ shows that the scale used for the forces does not affect the height of any joint of the polygon, which depends only on the scale according to which x or the horizontal span is drawn.

Let the line $P P^1$ be drawn parallel to $B C$, and let P^1 be the pole of a new equilibrium polygon. The pole distance will remain the same as before, but the pole distance represents the horizontal component of the stress in the closing line, and it has already been shown that if H remains the same v cannot vary. Hence, any movement of the pole parallel to the load line does not change the vertical dimensions of the equilibrium polygon, these dimensions varying only with the horizontal stress H and in the inverse ratio.

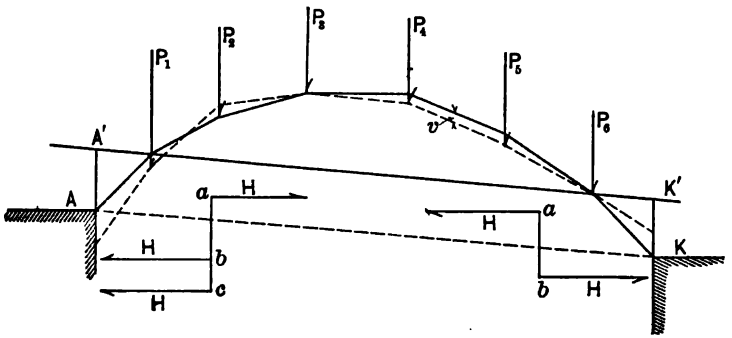
Fig. 3 shows that if the pole be moved parallel to the load line the direction of the closing line will be changed accordingly, but without changing the vertical dimensions of the polygonal frame. In this procedure the joints of the frame move in vertical lines, *i.e.*, parallel to the load line.

It is important to observe that the reaction at each end of a simply supported frame, when the end supports are not in the same horizontal line, is precisely the same as for a straight horizontal beam, each such reaction however being the algebraic sum of the vertical components of stress in the inclined closing line and the adjoining end member of the frame. This algebraic sum is the numerical difference at the lower point of support and the numerical sum at the higher. It is well also to observe that if the equilibrium polygon or frame were inverted so as to make the stress in the closing-line compression and the stresses in the sides or bars of the frame tension, the reverse would take place, *i.e.*, the reaction at the lower point of support would be the numerical sum of the vertical components of stresses in the closing line and the end member of the frame, and the difference of those quantities at the higher support.

The preceding constructions and diagrams show that when the end supports of a polygonal frame lie in the same horizontal line the closing line must be horizontal. As a general statement the closing line will coincide with or be parallel to the line connecting the two end supporting points. As every arch rib is a polygonal frame for some load, the closing line will always either connect the points of end supports of such a rib or be parallel to the line so formed.

Equilibrium Polygon for a Continuous Arch Rib with Fixed Ends.

The preceding constructions for polygonal frames or arch ribs with hinged ends may easily be adapted to ribs with fixed



ends, *i.e.*, ribs which are continuous with the supporting piers or abutments at their ends.

A rib with fixed ends differs from one with hinged ends only and precisely as a continuous beam differs from a non-continuous beam simply supported at its ends. In the analysis which establishes the Theorem of Three Moments it is shown that a continuous beam is nothing more than a simple non-continuous beam with a bending moment impressed upon each end. When these end moments are equal and opposite in sign the reactions at the two ends of the span remain precisely the same as if the beam were non-continuous and simply supported at its ends. If as in the general case the end moments are unequal, the simple non-continuous reactions are changed, one being less and the other greater by an amount found by dividing the algebraic difference of the end moments by the length of span.

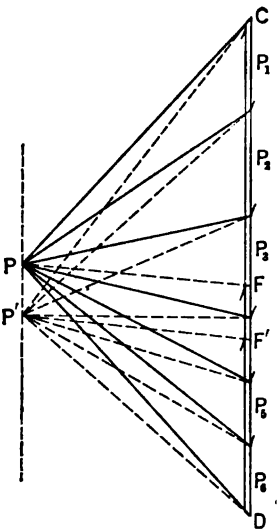


FIG. 4.

The full line polygon in Fig. 4 represents a polygonal frame with hinged ends, the sides of the polygonal frame being drawn in accordance with the full line force polygon, P being the pole. If H is the horizontal component of stress in the closing line $A\,K$, let two equal and opposite couples, each having the force H and lever arm $a\,b$, be impressed upon the two ends of the frame at A and K. As the force H of the couples is horizontal, the lever arm $a\,b$ is vertical. Since these couples are equal and opposite in sign the reactions at the two ends of the polygonal frame will be unchanged; but the closing line will be raised to $A^1\,K^1$, each lower H of the couple neutralizing the equal horizontal reactions at A and K. The vertical distance $A\,A^1 = K\,K^1 = a\,b$. The horizontal stress H will now be the horizontal component of the stress T acting along the new closing line $A^1\,K^1$.

As the reactions are unchanged by the action of these couples, the force polygon will also remain unchanged, $C\,F$ being the reaction at A and $F\,D$ the reaction at K.

If, however, the moment applied to the left end of the polygonal frame, for instance, is $M = H \times a\,c$, *i.e.*, larger than the moment applied at K, the partial moment $H \times b\,c$ will act to increase the reaction at A and decrease that at K by an amount equal to the moment $H \times b\,c$ divided by the horizontal span length between A and K and supposed to be represented by $F\,F^1$ in the force polygon. The closing line must be parallel to $A\,K$, the line joining the points of support; therefore, the line $F^1\,P^1$ in the force polygon is drawn parallel to $A\,K$ or $P\,F$. As the horizontal component of stress H is supposed to remain unchanged, the new pole P^1 will be at the intersection of a vertical line drawn through P^1, *i.e.*, parallel to $C\,D$, and the line drawn from F^1 parallel to the closing line.

If rays be drawn from P^1 to the various load points on $C\,D$, the directions of the sides of the new polygonal frame shown by broken lines in the upper part of Fig. 4 may be at once drawn. This new polygonal frame must have its position fixed vertically in reference to the new closing line $A^1\,K^1$ by the conditions of any given problem, as will be shown in a subsequent article where the fixed end elastic rib is graphically treated. Those con-

ditions may also be and in the general case are such as to change the pole distance and the horizontal component of stress H. The preceding construction, however, illustrates the kind of change introduced in the graphical construction by conditions of fixedness at the ends of the rib. The fixed moments in general not only raise the closing line of the force polygon parallel to itself, but make the value of the horizontal stress H differ from its hinged end value, while at the same time the reactions are changed as in the case of a continuous beam. Finally, it is evident from the construction in Fig. 4 that the moment ordinates in the case of a fixed end rib will be measured (positive and negative) above and below the closing line, the end ordinates measured from A^1 and K^1 to the ends of the polygonal frame representing the end bending moments.

Art. 82.—Actual Bending Moments in Arch Ribs.

It has been shown that if an arch rib or a frame has centre lines parallel to the sides of the equilibrium polygon for the loads which the rib or frame carries, no member of that rib or frame will be subjected to bending. This, however, is not the

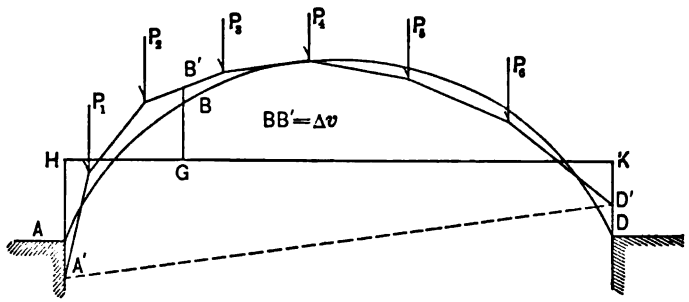


FIG. 5.

usual condition. In general the sides of the equilibrium polygon or the polygonal frame do not coincide with the centre line of the rib. In Fig. 5 the curved line $A\ B\ D$ is supposed to be the centre line of the arch rib, while the polygon $A^1\ B^1\ D^1$ is supposed

to be the true equilibrium polygon or polygonal frame for the system of vertical loads shown.

The line $H\,K$ is supposed to be the line of action of the horizontal stress H which is the constant horizontal component of the stresses in the sides of the true equilibrium polygon; and it is also the constant horizontal component of the resultant stress in the equilibrium polygon represented by the centre line of the arch produced by the loading for which the rib is a true equilibrium polygon or polygonal frame.

If B is any imaginary joint in the arch rib and if B^1 is the point in the polygonal frame immediately above it, then will $B\,G$ drawn vertically to $H\,K$ be the moment ordinate for the arch rib considered as a true equilibrium polygon for its proper load, while $B^1\,G$ will be the moment ordinate for the true equilibrium polygon for the system of loads which the arch rib must carry. That being the case the vertical intercept $B\,B^1$ must represent the actual bending moment in the arch rib at the joint B, as it is that part of the total moment ordinate not covered (*i.e.*, not canceled) by the ordinate of the rib itself. Expressed analytically, if M_b is the moment ordinate for the arch rib and if M_a be the moment ordinate for the true equilibrium polygon, both drawn vertically from $H\,K$, then will the bending moment in the rib, due to the load which the rib must actually carry, be:

$$M = M_a - M_b \quad . \quad . \quad . \quad . \quad . \quad . \quad 1.$$

This general equation may be written for any joint in the arch rib. Hence, the bending moment at any point or joint of an arch rib will be represented by the vertical intercept between the centre line of the rib and the true equilibrium polygon, the actual moment being computed by multiplying the horizontal stress H in the closing line $H\,K$ by the vertical intercept $B^1\,B = \Delta\,v$, or:

$$M = H \times B\,B = H\,\Delta\,v \quad . \quad . \quad . \quad . \quad 2.$$

This result may also be established, and perhaps more simply, in the following manner, it being supposed that the true

equilibrium polygon $A^1 B^1 D^1$ has been established for any given case whether the ends of the rib be fixed or hinged. Again, let B be any joint in the arch rib, while $G B$ and $G B^1$ are the moment ordinates for the rib and the polygonal frame, respectively. The ray in the force polygon parallel to that side of the true polygonal frame lying between the two loads P_2 and P_3 will represent the resultant force or stress in that side or member of the frame. Furthermore, the force represented by that ray is the resultant of all the forces or loads acting on the rib on one side of the vertical line $G B$. Let that resultant force be resolved into its vertical and horizontal components at the point B^1 in the frame vertically over B. As the vertical component acts in a vertical line passing through B it will have no moment about the latter point, but the horizontal component H will have a lever arm equal to $B B^1$. Hence the bending moment in the rib at the point or joint B will be:

$$M = H \times B B^1 = H \Delta v \quad . \quad . \quad . \quad . \quad 3.$$

If Δv represent in general the vertical intercept between the centre line of the arch rib and the true polygonal frame, the bending moment will always be represented by the equation:

$$M = \pm H \Delta v \quad . \quad . \quad . \quad . \quad . \quad 4.$$

The double sign is used because in the case of fixed end ribs the bending moment may be either positive or negative. If the rib has hinged ends the moment will evidently have one sign only.

The preceding considerations have been set forth so as to be perfectly general, *i.e.*, the principle established applies to ribs with either fixed or hinged ends, or with one end fixed and one end hinged. Furthermore, the reasoning by which Eqs. 3 and 4 are established holds true for any inclined loads as well as for vertical loads. Those equations, therefore, are applicable to any loadings whatever.

Art. 83.—General Formulæ for Flexure.

The following general formulæ* of the common theory of flexure will be required in the treatment of the elastic arch rib to be developed in the succeeding articles.

Let S denote the total shearing stress at any section of a rib or beam loaded vertically, n an indefinitely short part of the beam or rib, P any applied load or external force, M the bending moment, and D the vertical deflection, while D_h is the horizontal deflection at the same section; then the six general equations of flexure, some of which are used in the graphical treatment of arch ribs, are:

$$S = \Sigma P,$$

$$M = \Sigma P x,$$

$$u = \frac{M}{E I},$$

$$\Sigma n u = \sum \frac{n M}{E I},$$

$$D = \Sigma n u x = \sum \frac{n M x}{E I},$$

$$D_h = \Sigma n u y = \sum \frac{n M y}{E I}.$$

It will be recognized that $u = \frac{M}{E I}$ is the reciprocal of the radius of curvature due to bending in the case of a straight horizontal beam.

If the beam is originally straight and parallel to the axis of x, n becomes $d x$ and $y = 0$.

As usual, E and I represent the coefficient of elasticity and moment of inertia of the cross-section, respectively.

The limits of the summations are the section considered and any section of reference. The quantities S, M, u and D then

* Established in the author's "Elasticity and Resistance of Materials," Art. 18.

refer to that portion of the beam over which the summation extends.

One important deduction is to be drawn from the above equations, or rather from the second and fifth of them. It is seen from these two equations that $\frac{n\,M}{E\,I}$ stands in the same relation to D that P does to M. Consequently if $\frac{n\,M}{E\,I}$ be put in the place of P in any graphical construction, D will be represented in the place of M. If, therefore, an "equilibrium polygon" be constructed for any span by taking $\frac{n\,M}{E\,I}$ as loads instead of P, the vertical ordinates of the polygon will represent the deflections at the sections located by the corresponding values of x.

This polygon may be called the "deflection polygon," and its construction plays an important part in the determination of the true equilibrium polygon for an arch rib.

Art. 84.—Arch Rib with Ends Fixed.

Division of Rib into Parts $\Delta\,L$.

The ends of an arch rib or beam are considered fixed when the angles formed by their centre lines with any invariable line at the fixed sections do not change under any applied load.

Let $B\,A\,D$, Plate IV, be the centre line of an arch rib fixed at the points B and D. This line may be any curve, but for convenience a circular curve has been drawn. In the demonstration no attention whatever is given to the character of the curve, hence the treatment is equally applicable to any other curve.

In the present case the centre line of the rib is divided into the parts n determined by the intersections of the centre lines of the spandrel columns with that of the rib. These parts may sometimes be made of constant length, although there is no special advantage in that procedure.

Calling the entire length of the rib L, in all that follows, $\Delta\,L$ will be written in the place of n.

The division of the arch rib into the parts $\Delta\,L$ is a matter

of much importance in the actual application of this graphical method of treatment. As indicated above, if the load of the superstructure is transmitted to the rib through spandrel piers or columns it is convenient to make the length ΔL equal to the distance between the centres of those spandrel piers or columns measured along the centre line of the rib. If this distance should be too great it can be subdivided, but that is seldom desirable. This operation will in general make the parts ΔL of unequal length. Furthermore, the moment of inertia I is variable, having its smallest value at the crown and greatly increasing values in passing towards the ends of the span, especially with fixed ends as in the present instance. These conditions make the ratio $\frac{\Delta L}{I}$ a variable quantity in the equations of the preceding article.

At first sight the summations of that ratio may appear formidable, and efforts have been made to simplify that operation by so taking ΔL as to make the fraction $\frac{\Delta L}{I}$ a constant quantity. This procedure with the greatly varying values of I used in elastic arches built at the present time is highly objectionable and should be avoided. The rather involved operation of determining the lengths ΔL requisite to make the ratio $\frac{\Delta L}{I}$ constant prevents any saving of labor in the whole operation of numerical application, if indeed it does not add to the labors of computation, but besides that it has the fatal objection of making the points in the equilibrium polygon or line of resistance of the arch rib closer together at the crown than is necessary and much farther apart throughout the haunch of the arch and near the springing joint than is permissible. Furthermore, when the loads are transmitted to the arch rib through spandrel piers or columns great complication is introduced by disregarding the natural divisions of the rib determined by the points of loading at those spandrel columns. Each rib therefore will be divided, as indicated above, into parts determined by the centre lines of the spandrel columns.

It should be carefully observed that the moments M and the moments of inertia of cross-section I as well as other quantities used in this treatment are to be taken at points midway between the points of application of two consecutive loadings. This is shown on Plate IV, where b_0 is midway between the points of application of B and P_1; b_1 midway between P_1 and P_2, and so on throughout the rib. This operation is tacitly based on the assumption that the bending moment, moment of inertia, and thrust are each constant throughout their part ΔL of the rib. This clearly cannot be strictly true except when the points of application of each consecutive pair of loads are so near together as to constitute consecutive points of the curve. The assumption of finite values of ΔL necessary for actual computations is, however, sufficiently near the exact truth for all ordinary purposes.

The points in the axis of the rib midway between the points of application of the loading, *i.e.*, b_0, b_1, b_2, etc., are the imaginary normal joints of the rib at which the thrust, bending moment, and greatest intensities of stress are to be found in the process of designing the structure or of making such investigation in an existing elastic arch as may be necessary to determine its safe carrying capacity. The distance between these imaginary joints as $b_0\ b_1$, $b_1\ b_2$, $b_2\ b_3$, etc., must not be confused with the parts ΔL which are the distances along the centre line of the rib between the points of application of the loading.

General Conditions for Arch Rib with Ends Fixed.

The piers or points of fixedness are supposed to be immovable, whatever may be the distribution or amount of the load; and if that is the case, the summation of the direct strains of tension and compression at any given distance from the neutral axis of the rib, taken throughout its whole length $B\ A\ D$, must be equal to zero. If that distance is unity there results, for one condition, the equation:

$$\sum_D^B u\,\Delta L = \sum_D^B \frac{M\,\Delta L}{E\,I} = 0 \quad . \quad . \quad . \quad 5.$$

It will be necessary to find the successive numerical values of the second member of Eq. 5 by the aid of the data given for any particular case, the parts ΔL of the rib being determined in the manner described above.

It will next be necessary to establish values of the product EI corresponding to each of the parts ΔL. If the rib is of a single material like steel, E will have the value of the modulus of elasticity for that material, commonly taken at 30,000,000 lbs., although 29,000,000 lbs. is a better value for a built-up rib of steel shapes.

If the arch rib is built of two materials, as in the case of a reinforced concrete arch, the quantity EI must be displaced by the quantity $EI + E_2 I_2$, EI being the product of the modulus of elasticity of concrete multiplied by the moment of inertia of the normal section of the concrete at any point, while $E_2 I_2$ will represent the product of the modulus of elasticity of steel multiplied by the moment of inertia of the normal section of the steel part of the rib at any given point.

In either case if the work is one of design, tentative dimensions of cross-section of the rib corresponding to each part ΔL must be assumed before Eq. 5 can be applied. The bending moments M, also at first tentative, will be measured from the graphical diagrams used in this method.

Eq. 5 is that one of the equations of condition which is based upon the fixedness of the ends of the rib, and there are two others to be employed.

As the points B and D, Plate IV, are fixed in position, the deflections, both horizontal and vertical, taken between those points must each be equal to 0. It has been shown in the preceding article that the vertical deflection of any point in the rib, due to bending only, is:

$$D = \sum \frac{M x \Delta L}{EI} \quad . \quad . \quad . \quad . \quad . \quad 6.$$

The same general demonstration shows that the horizontal deflection due to bending has the same general form:

$$D_h = \sum \frac{M y \Delta L}{E I} \quad . \quad . \quad . \quad . \quad . \quad . \quad 7.$$

When these summations extend from one end of the rib to the other, Plate IV, D and D_h, Eqs. 6 and 7 must each equal 0, since both those points are assumed to be fixed in position. These three equations of condition, Eqs. 5, 6, and 7, must be simultaneously fulfilled for the arch rib with fixed ends.

Inasmuch as an equilibrium polygon or polygonal frame may be constructed for any load whatever, it is clear that any polygonal frame, whether the sides be straight and of finite length or whether they are infinitely short, thus constituting curved frames for continuous loads, must be an equilibrium polygon for some loading. Furthermore, the vertical intercept between the polygonal frame and its closing line must represent the bending moment which would exist in an ordinary straight beam, with ends fixed or otherwise as the case may be, of the same span and carrying the same loads.

Hence the curve $B\ A\ D$ in Plate IV is an equilibrium polygon for its proper load, and let its corresponding moment ordinates measured from the horizontal closing line be represented by M_b. Again, let a_0^1, a_1^1, a_2^1, etc., be the true equilibrium polygon for the given load and with the end conditions assumed. Then will the actual moments to which the rib is subjected be represented by the vertical intercepts between the true polygonal frame and the centre line of the rib; and let this actual moment intercept be represented by M. If the moment ordinates of the true equilibrium polygon, measured from its closing line (the same from which M_b is drawn), be represented by M_a:

$$M = M_a - M_b \quad . \quad . \quad . \quad . \quad . \quad . \quad 8.$$

Then by Eq. 5:

$$\sum_D^B \frac{M \Delta L}{E I} = \frac{1}{E} \sum_D^B \frac{M_a \Delta L}{I} - \frac{1}{E} \sum_D^B \frac{M_b \Delta L}{I} = 0 \quad . \quad 9.$$

or:

$$\sum_D^B \frac{M_a \Delta L}{I} = \sum_D^B \frac{M_b \Delta L}{I} \quad . \quad . \quad . \quad . \quad 10.$$

Eq. 9 requires only that the first and second terms of its second member shall be equal to each other, and that equality is most simply and conveniently attained by making each member equal to zero. Hence the condition of fixedness for the ends of the rib may be put in the form:

$$\sum_D^B \frac{M_a \Delta L}{I} = 0 = \sum_D^B \frac{M_b \Delta L}{I} \quad . \quad . \quad . \quad 11.$$

Similarly Eqs. 6 and 8 will give:

$$\sum_D^B \frac{M x \Delta L}{EI} = \frac{I}{E}\left(\sum_D^B \frac{M_a x \Delta L}{I} - \sum_D^B \frac{M_b x \Delta L}{I}\right) = 0 \; . \; 12.$$

Eq. 12 expresses the condition, only, that the two terms in the parenthesis of the second member shall be equal to each other, and that equality is again most conveniently attained by making each term equal to zero. Therefore:

$$\sum_D^B \frac{M_a x \Delta L}{I} = 0 = \sum_D^B \frac{M_b x \Delta L}{I} \quad . \quad . \quad 13.$$

Finally, as has been stated already, the horizontal deflection, given by Eq. 7 of one end of the rib in reference to the other must be zero. By using Eqs. 7 and 8, therefore:

$$\sum_D^B \frac{M_a y \Delta L}{I} = \sum_D^B \frac{M_b y \Delta L}{I} \quad . \quad . \quad . \quad 14.$$

The equality indicated by Eq. 14 cannot be attained by making each member equal to zero, but it can be simply employed by constructing two trial polygons to determine the pole distance of the true equilibrium polygon or polygonal frame, as will presently be shown.

It must be kept in mind that M represents the actual moment to which the rib is subjected at any point and that it is proper to represent it as the difference between the two moment elements M_a and M_b. The application of these equations will be shown as they are needed in the course of the following graphical constructions.

Application to an Actual Arch Rib.

This graphical method will be applied to the same reinforced concrete rib and with the same loading as given in Art. 98 of Chapter VIII. Part of the same data used in that article will, however, be repeated here for convenience of reference. It will be observed that there are a few slight numerical differences in the two sets of data relating to the ΔL's as well as to the ordinates y and b of the imaginary joints and points of application of the loading to the centre line of the rib, but they are not material in their influence on the corresponding numerical results. As these data show, the span between the centres of the springing joints is 151.2 feet, while the rise of the centre line of the rib at the crown is 47 feet.

ΔL	Sq. Ft. Total Section. A	Sq. Ft. Steel Section. A_s
$\Delta L_{13} = \Delta L_0 = 4.0$ ft.	$A_0 = 24.4$	$A_{s0} = .125$
$\Delta L_{12} = \Delta L_1 = 19.9$ "	$A'_0 = 24.2$	$A'_{s0} = .125$
$\Delta L_{11} = \Delta L_2 = 16.5$ "	$A_1 = 22.8$	$A_{s1} = .1$
$\Delta L_{10} = \Delta L_3 = 14.6$ "	$A_2 = 19.2$	$A_{s2} = .1$
$\Delta L_9 = \Delta L_4 = 13.3$ "	$A_3 = 16.6$	$A_{s3} = .08$
$\Delta L_8 = \Delta L_5 = 12.7$ "	$A_4 = 14.3$	$A_{s4} = .08$
$\Delta L_7 = \Delta L_6 = 12.3$ "	$A_5 = 11.7$	$A_{s5} = .06$
93.3 "	$A_6 = 11.4$	$A_{s6} = .06$

Moment of Inertia.	Depth of Rib.	Feet.
$I_0 = 135.8$	$h_0 = h_{14} = 7.5$ ft.	$x'_0 = 1.05$ ft.
$I'_0 = 132.2$	$h'_0 = h_{13} = 7.46$ "	$x_1 = 8$ "
$I_1 = 106.8$	$h_1 = h_{12} = 7$ "	$x_2 = 20.4$ "
$I_2 = 65.86$	$h_2 = h_{11} = 5.9$ "	$x_3 = 32.4$ "
$I_3 = 41.9$	$h_3 = h_{10} = 5.1$ "	$x_4 = 44.8$ "
$I_4 = 27.35$	$h_4 = h_9 = 4.4$ "	$x_5 = 57.2$ "
$I_5 = 14.65$	$h_5 = h_8 = 3.6$ "	$x_6 = 69.6$ "
$I_6 = 13.53$	$h_6 = h_7 = 3.5$ "	$x_7 = 81.6$ "
		$x_8 = 94$ "
		$x_9 = 106.4$ "
		$x_{10} = 118.8$ "
		$x_{11} = 130.8$ "
		$x_{12} = 143.2$ "
		$x_{13} = 150.15$ "
		$x_{14} = 151.2$ " (span)

Feet.		Feet.		Feet.	
y'_0 = 1.6 ft.		a_1 = 2.1 ft.		b_1 = 3.4 ft.	
y_1 = 11.3 "		a_2 = 14.35 "		b_2 = 18.9 "	
y_2 = 24.7 "		a_3 = 26.6 "		b_3 = 30.0 "	
y_3 = 34.1 "		a_4 = 38.85 "		b_4 = 37.8 "	
y_4 = 40.8 "		a_5 = 51.1 "		b_5 = 43.1 "	
y_5 = 44.8 "		a_6 = 63.35 "		b_6 = 46.1 "	
y_6 = 46.8 "		a_7 = 75.6 "		b_7 = 47.0 "	(Rise)
y_7 = 46.8 "		a_8 = 87.85 "		b_8 = 46.0 "	
y_8 = 44.8 "		a_9 = 100.1 "		b_9 = 43.0 "	
y_9 = 40.8 "		a_{10} = 112.35 "		b_{10} = 37.8 "	
y_{10} = 34.1 "		a_{11} = 124.6 "		b_{11} = 29.8 "	
y_{11} = 24.7 "		a_{12} = 136.85 "		b_{12} = 18.4 "	
y_{12} = 11.3 "		a_{13} = 149.1 "		b_{13} = 3.2 "	
y_{13} = 1.6 "					
y_{14} = 0 "					

The uniform width of the rib is 3.25 ft. The depth of rib at the crown is 3.5 feet and 7.5 feet at the springing joint. There are thirteen points of application of dead and live load. It will be assumed that the arch rib carries in addition to its own weight six live loads applied at the six points of application of P_1, P_2, P_3, P_4, P_5, P_6, on the left half of the structure. The six live and thirteen dead loads applied at the points indicated will be as follows:

Dead Load.	Live Load.	Dead Load.
P_1 = 75,600 lbs.	24,000 lbs.	P_8 = 36,500 lbs.
P_2 = 53,300 "	24,500 "	P_9 = 44,000 "
P_3 = 49,800 "	24,500 "	P_{10} = 47,400 "
P_4 = 47,400 "	24,500 "	P_{11} = 49,800 "
P_5 = 44,000 "	24,500 "	P_{12} = 53,300 "
P_6 = 36,500 "	24,500 "	P_{13} = 75,600 "
P_7 = 36,500 "		

This live loading of uniform density covering a little less than half the span will produce nearly, if not quite, the maximum bending moment to which any part of the rib can be subjected, and it will be shown further on that the results may be used in obtaining those which belong to the condition of the entire span covered by the moving load.

It will generally be advisable to construct a line of resistance for the dead load only. After the completion of the construction

which is to be made for the loads described, the method to be followed for the dead load alone, or for the live load on any part of the span whatever, or on the whole span, will easily be apparent.

Construction of First Trial Polygon.

The first condition to be applied is that expressed by Eq. 11:

$$\sum_D^B \frac{M \cdot \Delta L}{I} = 0 \quad . \quad . \quad . \quad . \quad . \quad 15.$$

The ΔL's and I's for the different joints, b_0, b_1, b_2, etc., of the rib shown on Plate IV, are given in the preceding tabular statement of data. The trial values of the quantities M_a are to be determined by a trial polygon. A vertical line, 1–14, drawn to the left of B is to be taken as the load line of the trial polygon. The force scale used is shown on the plate, *i.e.*, 60,000 lbs. to the inch. The load P_1 is represented by the segment 1–2 of the load line; the load P_2 by the segment 2–3; the load P_3 by the segment 3–4, and so on for the entire rib to the lower extremity 14 of the load line, the loads on the right part of the arch being represented by the segments of the lower part of the load line.

If any convenient point C^1 be taken as the trial pole, the normal $C^1 L^1$ drawn from C^1 to the load line will be the pole distance. After the rays are drawn from that pole to the various points indicated on the load line, the trial funicular or equilibrium polygon $E C_1 F$ may at once be constructed. Starting from C_1 the first side is drawn from that point parallel to the ray $C^1 - 7$ to its intersection with the line of action of P_6 prolonged; from that point will be drawn the next side of the polygon parallel to the ray $C^1 - 6$ until it intersects the line of action of P_5 prolonged, and so on until the completion of the sides $C_1 a_6 a_5 \ldots a_0 E$. The right-hand part of the same polygonal frame $C_1 a_7 a_8 \ldots a_{13} F$ is constructed by the same process, using the rays drawn from C^1 to the points 8, 9, 10, . . . 14. The complete trial polygon $E C_1 F$ is thus found. If the ends of this rib were hinged the closing line would evidently be drawn from E to F. Inasmuch, however, as the ends are fixed, corresponding moments of proper

sign will be imposed upon the ends of the arch showing that the closing line must occupy some position as $H\,K$ between $E\,F$ and the crown C_1. If it be assumed that $H\,K$ is the closing line sought, the lengths of vertical ordinates drawn from $H\,K$ downward to the rib at and in the vicinity of the ends of the span and upward from the same line to the rib at and in the vicinity of the crown will be in proportion to the bending moments at the points where these vertical ordinates intersect the rib, *i.e.*, these vertical intercepts between the closing line and the polygonal frame will represent the bending moments which would exist in a straight beam with fixed ends of the same span as the arch and carrying the same loads. As shown on the plate the vertical intercept $c_6\,a_6$ will represent the value of M_a in the straight beam at a_6; $c_1\,a_1$ will represent the bending moment in the same beam at a_1, and so on throughout the rib. Furthermore, the actual bending moments are found by simply multiplying these vertical intercepts by the force represented by the pole distance of the force polygon.

The first condition to be fulfilled, represented by Eq. 11, consists in determining the sum of all the quantities $\frac{M_a\,\Delta L}{I}$ throughout the rib, at the same time so locating the closing line $H\,K$ as to make that sum zero.

In Plate IV, $H\,K$ has been located so as to make $c_0\,a_0 = 2.4$ ins. on the original drawing, at the same time making $c_{13}\,a_{13} = 1.97$ ins., $c_6\,a_6 = .49$ ins., and $c_7\,a_7 = .43$ ins. Calling the intercepts below $H\,K$ negative and those above positive there will result:

$$\text{Plus } \sum_D^B \frac{M_a\,\Delta L}{I} = + .06 + .34 + .45 + .39 + .24 + .01 = + 1.49.$$

$$\text{Minus } \sum_D^B \frac{M_a\,\Delta L}{I} = - .07 - .35 - .24 - .11 - .13 - .22 - .29 - .06 = - 1.47.$$

The difference between these plus and negative products is so near zero that the trial closing line $H\,K$ shown on the plate may be accepted as substantially correct.

As already explained, precisely the same procedure must be followed to locate properly the closing line $H^1 K^1$ for the centre line $B\,A\,D$ of the rib considered as an equilibrium polygon or a polygonal frame for some load which it is not necessary to determine. Inasmuch as the centre line of the rib is symmetrical in reference to a vertical line through its crown it is clear that the system of loading belonging to it will also be symmetrical in reference to the same vertical line, making the bending moments M_b at all points equally symmetrical. The closing line $H^1 K^1$ must, therefore, be horizontal. After proper trial it was found that the location indicated on the plate would satisfy the prescribed condition. On the original drawing the line $c^1_0 b_0$ was 3.81 ins.; $c^1_1 b_1$, 2.81 ins.; $c^1_6 b_6$, .73 ins.; $c^1_5 b_5$, .53 ins.; etc. Making the intercepts above $H^1 K^1$ positive and those below negative, the following sums of products will be found by scaling the intercepts from the drawing and taking the corresponding ΔL's and I's from the statement of data given above:

$$\text{Plus} \sum_D^B \frac{M_b\,\Delta L}{I} = 2\,(.06 + .46 + .66) = +\,2.36.$$

$$\text{Minus} \sum_D^B \frac{M_b\,\Delta L}{I} = -\,2\,(.11 + .53 + .37 + .18) = -\,2.38.$$

These positive and negative sums are so nearly equal numerically that the position of $H^1 K^1$ shown on the plate may be considered substantially correct.

The preceding operations fulfill the condition for the trial polygons $E\,C_1\,F$ and $B\,A\,D$, as shown by Eq. 11, but the condition expressed by Eq. 13 must be concurrently fulfilled.

In order to meet the requirements of the latter equation, each of the terms $\frac{M_a\,\Delta L}{I}$ and $\frac{M_b\,\Delta L}{I}$ used in the preceding plus and minus summations is to be multipled by the corresponding value of x, so as to make $\sum_D^B \frac{M_a x\,\Delta L}{I} = \sum_D^B \frac{M_b x\,\Delta L}{I}$. Obviously as that summation represents the vertical deflection of one end of the rib in reference to the other, it is a matter of no

consequence whether x be measured from one end of the rib or the other; it will be measured from the right-hand end, D. It is required only that the two summations shall be equal, and that equality is conveniently secured by making each sum equal to zero. The summation $\sum_D^B \frac{M_a x \Delta L}{I} = 0$ will first be considered.

Inasmuch as the horizontal distance between two consecutive loads is constant, *i.e.*, 12.25 ft., it will simplify computations to express the horizontal co-ordinate x in terms of that constant distance. The point b_{13} is horizontally distant 1.1 ft. from D; hence $\frac{1.1}{12.25} = .09$ will be used as the corresponding factor of $\frac{M_a x \Delta L}{I}$. In the same way b_{12} is horizontally distant 8.1 ft. from D; hence, $\frac{8.1}{12.25} = .66$ will take the place of x in the corresponding term $\frac{M_a x \Delta L}{I}$. Similarly 1.66 will take the place of x in the term $\frac{M_a x \Delta L}{I}$ for the point b_{11}, and 2.66 for b_{10}, etc. Writing out the positive summations representing that part of the polygon above the closing line $H K$ there will result:

$$\textit{Plus terms of } \sum_D^B \frac{M_a x \Delta L}{I} = + 12.25\,(.01 \times 3.66 + .24 \times 4.66 + .39 \times 5.66 + .45 \times 6.66 + .34 \times 7.66 + .06 \times 8.66) = + 12.25 \times 9.4836 = + 116.17.$$

In the same manner:

$$\textit{Minus terms of } \sum_D^B \frac{M_a x \Delta L}{I} = - 12.25\,(.06 \times .09 + .29 \times .66 + .22 \times 1.66 + .13 \times 2.66 + .11 \times 9.66 + .24 \times 10.66 + .35 \times 11.66 + .07 \times 12.23) = - 12.25 \times 9.516 = - 116.571.$$

The sum of these two summations is $- .4$, which is so near zero that the closing line $H K$ may be accepted as fulfilling the conditions of both Eqs. 11 and 13 without material error.

The fulfilment of the condition $\sum_D^B \frac{M_b x \Delta L}{I} = 0$ for the polygon $B\ A\ D$ follows at once as a consequence of the condition $\sum_D^B \frac{M_b \Delta L}{I} = 0$. This will be clear if it be observed that the quantities $\frac{M_b \Delta L}{I}$ are symmetrically located in reference to the vertical line through the crown A and that the members of each symmetrical pair have each the same value. These conditions make the sums of the positive and negative members of the summation $\sum_D^B \frac{M_b x \Delta L}{I}$ equal to each other, and hence their aggregate sum equal to zero. The closing line $H^1\ K^1$ shown on Plate IV thus fulfils the requirements of Eq. 13 for the polygonal frame $B\ A\ D$.

The preceding operations have determined the location of the two closing lines for the moments M_a and M_b so as to indicate those portions of the rib subjected to positive or negative moments, but the pole distance of the true equilibrium polygon yet remains to be determined by the fulfilment of the final condition:

$$\sum_D^B \frac{M_a y \Delta L}{I} = \sum_D^B \frac{M_b y \Delta L}{I} \quad . \quad . \quad . \quad 16.$$

It has already been shown in the preceding article that if the quantities $\frac{M_a \Delta L}{I}$ and $\frac{M_b \Delta L}{I}$ be considered as loads, the summation of the corresponding products $\sum_D^B \frac{M_a y \Delta L}{I}$ and $\sum_D^B \frac{M_b y \Delta L}{I}$ will be the moment ordinates (horizontal in this case) of equilibrium polygons or polygonal frames which will represent the horizontal deflections of a beam or of the rib under treatment.

It will thus be necessary to construct two equilibrium poly-

gons, one by using the series of expressions $\frac{M_a \Delta L}{I}$ as horizontal loads acting at the vertical distances y from the line $B\,D$ and the other by using the series of quantities $\frac{M_b \Delta L}{I}$ as horizontal loads acting at the same vertical distances y above the line $B\,D$. The figure at the right of the arch on Plate IV, with the title "Deflection Polygons," illustrates this operation.

The horizontal line $B\,D$ is extended to the pole X from which the vertical line $A^1\,X$ is drawn upward, the point A^1 being projected horizontally from A the crown of the arch. In accordance with Eq. 14, $\sum_D^B \frac{M_a y \Delta L}{I}$ must be equal to $\sum_D^B \frac{M_b y \Delta L}{I}$, each of these summations representing the horizontal deflection of the rib taken between the limits indicated. These quantities, in fact, represent the horizontal deflection of a straight vertical beam with length $X\,A^1$ subjected to the bending moments M_a and M_b at points at the same vertical distances y above the horizontal line $B\,D$.

Inasmuch as the joints b_{13} and b_0, b_{12}, b_1, etc., are at the same vertical distance y above $B\,D$ in pairs, it will simplify the construction to take the sum of each such pair of moments instead of treating each one separately. Hence, taking the horizontal line through A^1 as a load line, $A^1\,t$ is laid off to represent $\frac{M_a \Delta L}{I}$ for the two joints b_{13} and b_0, that distance being $-.13$ as laid down by a scale of 10 units to the inch. In a similar manner $t\,t_1 = -.64$ represents the sum of the two quantities $\frac{M_a \Delta L}{I}$ for the two joints b_{12} and b_1; $t_1\,t_2 = -.46$ represents the sum of the same quantities for the joints b_{11} and b_2; $t_2\,t_3 = -.24$ represents the sum of the same two quantities for the joints b_{10} and b_3; $t_3\,t_4 = +.07$, measured in the opposite direction as it represents positive moments or moment ordinates measured upward from the closing line, represents the sum of the same quantities for the joints

b_9 and b_4; $t_4\, t_5 = +\ .58$ represents the sum of the same quantities for the joints b_8 and b_5; finally, $t_5\, A^1 = +\ .84$ represents the sum of the last pair of the series for the two joints b_7 and b_6. Rays are then drawn in the usual way from the pole X to the various points, $t, t_1, t_2, t_3, t_4, t_5$. The deflection polygon is then completed by drawing its sides parallel to the rays to their intersections with the horizontal lines projected over from joints b_{13}, b_{12}, b_{11}, b_{10}, etc. This operation gives the desired deflection polygon $X\, C^{11}$, d_1, d_2, d_3, d_4, d_5, d_6, and the point on the horizontal line through A^1 at the left extremity of l. This latter quantity l is the horizontal deflection of either end of the rib in reference to the other which would be produced by the moments M_a acting upon it at the joints indicated.

Construction of Second Equilibrium Polygon.

The corresponding deflection polygon for the rib $B\, A\, D$, shown on the right-hand side of $A^1\, X$, is drawn in precisely the same manner by using the moment ordinates M_b measured from $H^1\, K^1$, and normal to it, to the centre line of the rib $B\, A\, D$. The plus and minus values of $\frac{M_b\, \Delta\, L}{I}$ are shown on page 285.

In accordance with those values $A^1\, s = -\ .22$ (adding symmetrical pairs) for joints b_0 and b_{13}; $s\, s_1 = -\ 1.06$; $s_1\, s_2 = -\ .74$; $s_2\, s_3 = -\ .36$; $s_3\, s_4 = +\ .12$; $s_4\, s_5 = +\ .92$; $s_5\, s_6 = +\ 1.32$. Again, drawing rays from X to the points, s, s_1, s_2, s_3, s_4, etc., thus established, and fixing their intersections with horizontal lines drawn through the joints of the rib the deflection polygon $X, C,'' d^1_1, d^1_2, d^1_3, d^1_4, d^1_5, d^1_6$ is completed, giving l^1 as the horizontal deflection sought.

The deflection l is seen to be much smaller than l^1, showing that the moment ordinates M_a must be increased in the proportion of l to l^1, which is equivalent to decreasing the trial pole distance $C^1\, L^1$ in the proportion of l^1 to l. Hence in the "Proportional Scale," shown at the lower right-hand corner of Plate IV, lay off $Y\, M$ equal to l on the horizontal line through

Y and erect a vertical line through M. Then swing a radius equal to l^1 from Y as a centre till it cuts the vertical through M in N, and draw the indefinite line $Y\ N\ P$.

If $Y\ P$ be made equal to the trial pole distance $C^1\ L^1$ and if a vertical be dropped from P to O, $Y\ O$ will be the true pole distance. Hence draw $C^1\ L$ parallel to $H\ K$, then make the horizontal line $C\ L$ equal to $Y\ O$ in the Proportional Scale. The rays drawn from the true pole C will be parallel to the sides of the true polygonal frame or equilibrium polygon sought.

Any number of points in the true equilibrium polygon may readily be found by using the moment ordinates M_a and the Proportional Scale. Lay off $Y\ g_1$ in the latter equal to $c_1\ a_1$ and erect the vertical $g_1\ p_1$. Then make $c^1_1\ a^1_1$ equal to $Y\ p_1$ and a^1_1 will be a point in the side of the true polygonal frame parallel to the ray $C\ 2$. Similarly every other side of that frame may be drawn; or the true polygon may be completed by the aid of the rays from the true pole after one side has been determined in the manner shown. Both methods should be used to check results. The true frame E^1, a^1_0, a^1_1, a^1_2, etc., $a^1_{12}\ F^1$ is thus established.

Horizontal Stress H.

The pole distance $C\ L$ scales 4.52 inches; hence the horizontal thrust H, *i.e.*, the constant horizontal component of the resultant stress in each side of the polygonal frame whose line of action is $H^1\ K^1$, is:

$$H = 4.52 \times 60{,}000 = 271{,}200 \text{ lbs.}$$

If a horizontal tie were used to take up the thrust of the rib the tension in it would be, obviously, the above value of H.

Bending Moments.

The bending moment at any joint in the rib may now be found by simply multiplying the vertical intercept between the centre line of the rib and the true polygonal frame, at any joint,

by the above value of H. If the springing joints at B and D be included, the vertical intercepts and moments at the various joints indicated will be the following:

Joint.	Vertical Intercept.	Moment.
B	− 2.2 ft.	− 596,640 ft. lbs.
b_0	− 2.1 "	− 569,520 " "
b_1	− 2.7 "	− 732,240 " "
b_2	− 1.4 "	− 379,680 " "
b_3	.00"	0 " "
b_4	+ .9 "	+ 24,408 " "
b_5	+ 1.2 "	+ 325,440 " "
b_6	+ .7 "	+ 18,984 " "
b_7	− .1 "	− 2,712 " "
b_8	− .8 "	− 21,696 " "
b_9	− 1.1 "	− 298,320 " "
b_{10}	− .8 "	− 21,696 " "
b_{11}	+ .2 "	+ 5,412 " "
b_{12}	+ 2.2 "	+ 596,640 " "
b_{13}	+ 4.8 "	+ 1,301,760 " "
D	+ 5.0 "	+ 1,356,000 " "

These bending moments must be used in determining the greatest intensities of bending stresses either for a rib of a single material such as steel or concrete or for a combined material like reinforced concrete. The greatest bending intensities must then be algebraically added to the intensities due to direct thrust, in order to obtain the greatest resultant intensities at all joints.

Thrusts and Shears.

The lengths of the rays drawn from the true pole to the load line, as shown on Plate IV, will give the thrusts at the various joints of the rib, using the scale of 60,000 lbs. per linear inch. Although these thrusts are not accurately normal to the joints to which they correspond they may be so regarded without sensible error. The component of each of these thrusts parallel to the joint on which it acts will give the ransverse shear at each such joint. These tangential or shearing components are shown on the plate. The following tabulation gives the joints

and the corresponding shears and resultant thrusts for the fourteen joints of the rib:

Joint.	Resultant Thrust.	Shear.
b_0	530,000 lbs.	+ 9,500 lbs.
b_1	448,000 "	+ 6,000 "
b_2	387,000 "	+ 23,000 "
b_3	338,000 "	+ 18,500 "
b_4	301,000 "	+ 13,500 "
b_5	278,000 "	− 3,000 "
b_6	271,000 "	− 19,000 "
b_7	274,000 "	− 14,000 "
b_8	281,000 "	− 5,500 "
b_9	295,000 "	0 "
b_{10}	316,000 "	+ 14,000 "
b_{11}	344,000 "	+ 24,000 "
b_{12}	377,000 "	+ 49,000 "
b_{13}	434,000 "	+ 53,000 "

The plus and minus signs of the shears simply indicate the change of direction of the tangential component of the thrust as will be apparent by comparing the direction of the thrust at any joint with the direction of the centre line of the rib at the same point.

Vertical Reactions at Ends of Span.

The vertical reactions at the ends of the span B and D, respectively, can be scaled from the load line 1–14. $C\,L$ being horizontal, L–1 represents the reaction at B, while L–14 represents the reaction at D. Hence as L–14 scales 7.59 inches, the reaction at B is:

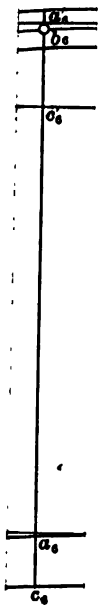
$$7.59 \times 60{,}000 = 455{,}400 \text{ lbs.}$$

Similarly as L–14 scales 5.7 inches, the reaction at D is:

$$5.7 \times 60{,}000 = 342{,}000 \text{ lbs.}$$

The sum of these two reactions is 797,400 pounds, which is, as closely as can be scaled, the total load on the rib.

C



Tr

Ends.

Art. 85.—Arch Rib with Hinged Ends.

An arch rib with hinged ends obviously has no constraining moment applied to those ends; the latter are free to take any direction in a vertical plane consistent with the state of stress produced in them by the given loading. If the ends are hinged by resting on steel or other metal pins the friction on the surfaces of the latter tends to resist any motion which the ends of the rib might otherwise have in consequence of the stresses in them, but this frictional resistance is usually ignored.

If the hinged condition of the ends is produced by a steel or other metal plate which is supposed to bend within its elastic limit to give the ends of the ribs the desired freedom to take any required change of direction, the resistance to bending of those metal plates is also a constraining tendency, but it is neglected in an analytical or graphical treatment of elastic ribs, as is the friction on the pins when used as hinges. It is feasible as a matter of analysis to take into account these constraining influences which may be considered secondary, but they are not of sufficient value to make it generally advisable to do so.

As the ends of the rib in this case are hinged and assumed free to take such change of direction as the conditions of stress in them may require, the condition $\sum_{D}^{B} \frac{M \Delta L}{EI} = 0$, expressing full constraint at the ends, does not hold. This means essentially that the bending moments at the hinged ends are zero.

In order to make the graphical constructions for this case the same rib and the same loading used in the preceding article will be taken. Hence the dimensions of the rib will also remain the same.

Although the change from fixed to hinged ends would ordinarily require some changes in the dimensions and moments of inertia of the normal cross-sections at and near the springing joints as well as in the percentage of steel, for the purposes of comparison of results those data as well as others will remain

unchanged. In the following construction therefore all numerical data required will be taken as established in the preceding article.

The ends of the centre line of the rib must remain at an invariable horizontal distance apart. This means that the horizontal deflection of either end of the centre line of the rib in reference to the other must be zero, and such a condition is expressed by the following equation:

$$\sum_{D}^{B} \frac{My\,\Delta L}{EI} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 17.$$

It must be observed that any vertical movement of one end of the span in reference to the other, as long as the span length remains unchanged, will not affect in any way whatever the stress conditions of the arch rib unless the elevation of one end in reference to the other is so great as to change materially the direction of the rib in reference to the applied vertical loading. The equation showing the vertical deflection of one end of the rib in reference to the other therefore cannot be used as an equation of conditions.

Taking any convenient point C, Plate V, as the pole for the trial polygon and erecting the vertical line 1–14 through B as a load line, the segment 7–8 of the latter (B being assumed as its middle point) represents the load P_7 at the crown A of the arch by a scale of 60,000 lbs. per inch on the original drawing. By the same scale 8–9 represents the load P_6; 9–10, the load P_5, . . . and 13–14, the load P_1. Similarly the loads P_8, P_9, . . . P_{10} are represented at the same scale by 7–6; 6–5 . . . and 2–1 on the upper part of the load line. Drawing rays from C the trial polygon $E\ a_0\ a_1\ a_2\ \ldots\ a_{13}\ F$ is constructed in the usual manner by making the sides of that polygon parallel to the rays, each side being drawn to the intersections with the lines of action prolonged of each two consecutive loads. The closing line in this trial polygon is EF as the ends of the rib are hinged, making the bending moments at these points zero. The ordinates $a_0\,c_0$, $a_1\,c_1$, $a_2\,c_2$, etc., represent or are proportional to the bending moments which would be produced in a straight beam

B D simply supported at its ends and carrying the loads resting on the rib, *i.e.*, they are the moment ordinates.

It has been shown above, in connection with the analysis in the preceding article, that the equation of condition to be fulfilled in this case is:

$$\sum_D^B \frac{My\,\Delta L}{EI} = \sum_D^B \frac{M_a y\,\Delta L}{EI} - \sum_D^B \frac{M_b y\,\Delta L}{EI} = 0, \text{ or,}$$

$$\sum_D^B \frac{M_a y\,\Delta L}{I} = \sum_D^B \frac{M_b y\,\Delta L}{I} \quad . \quad . \quad . \quad . \quad 18.$$

As in the preceding article, E being constant divides out of this equation.

The trial polygon *E C F*, Plate V, as constructed gives the desired moment ordinates M_a. As the rib has hinged ends the true closing line must pass through the centres of its hinges at *B* and *D*, the moment ordinates *M* being measured from it.

The deflection polygon must be drawn to fulfil the condition shown by Eq. 18. The quantities entering the sum of the series $\sum_D^B \frac{M_a\,\Delta L}{I}$ are shown in the following tabulation.

Joint	$\frac{\Delta L}{I}$	Moment Ordinate M_a	$\frac{M_a\,\Delta L}{I}$
a_0	.03	.1 in.	$A\,t_0 =$ 0.0
a_1	.19	.66 "	$t_0\,t_1 =$ .11
a_2	.25	1.49 "	$t_1\,t_2 =$ .34
a_3	.35	2.10 "	$t_2\,t_3 =$ .68
a_4	.49	2.50 "	$t_3\,t_4 =$ 1.16
a_5	.87	2.73 "	$t_4\,t_5 =$ 2.28
a_6	.91	2.78 "	$t_5\,t_6 =$ 2.49
a_7	.91	2.69 "	
a_8	.87	2.51 "	The above numerical values are
a_9	.49	2.22 "	each one half of
a_{10}	.35	1.80 "	a pair, having the
a_{11}	.25	1.25 "	same *y*.
a_{12}	.19	0.54 "	
a_{13}	.03	.10 "	

The deflection polygon for the first trial polygon *E C F* will be drawn by taking *A C* as a trial pole distance and using

the horizontal line drawn through A as the line on which the deflection loads or quantities $\frac{M_a \Delta L}{I}$ are laid off. The imaginary joints in the rib are symmetrically placed in pairs in reference to the crown A, each pair having the same vertical ordinate y. Hence the sum of the above quantities for each such pair will be used precisely as was done in the preceding article for the rib with fixed ends. Furthermore as the sum of all those quantities would exceed the length of $R\,A$ for the moments M_a and the length $S\,A$ for the moments M_b they will each be laid down at half full size.

The product $\frac{M_a \Delta L}{I}$ is so small for the joints b_0 and b_{12} that it is considered zero; the sum of the half products for the two joints b_1 and b_{12} is .11″ and is laid off from A to t_1. Similarly $t_1 t_2 = .34$ represents the sum of the same half products for the two joints b_2 and b_{11}. Proceeding in precisely the same way $t_2 t_3 = .68''$; $t_3 t_4 = 1.16''$; $t_4 t_5 = 2.28''$; and $t_5 t_6 = 2.49''$. Drawing the rays from C to the points t thus established the deflection polygon C, C'', d_1, d_2, d_3, d_4, d_5, d_6, is completed. The deflection distance l is then determined at half its full size.

The moment ordinates M_b are next to be treated in the same way. These ordinates are equal in pairs, each pair also having the same vertical ordinate y. By careful scaling from the original drawing the following tabulated values are found and computed:

Joint	$\frac{\Delta L}{I}$	Moment Ordinate M	$\frac{1}{4}\frac{M_b \Delta L}{I}$
a_0	.03	.19 in.	$A\,s_0 = 0.0$
a_1	.19	1.17 "	$s_0 s_1 = .11$
a_2	.25	2.49 "	$s_1 s_2 = .31$
a_3	.35	3.43 "	$s_2 s_3 = .60$
a_4	.49	4.07 "	$s_3 s_4 = 1.00$
a_5	.87	4.48 "	$s_4 s_5 = 1.95$
a_6	.91	4.68 "	$s_5 s_6 = 2.13$

One-fourth of the quantities $\frac{M_b \Delta L}{I}$ only are used in this case, making the deflections one-quarter their full size. The quantity for joints b_0 and b_{13} is so small as not to be appreciable at the scale used, hence it is considered zero. $A\ s_1$ is then laid down as .11″, $s_1 s_2 = .31''$; $s_2 s_3 = .6''$; $s_3 s_4 = 1''$; $s_4 s_5 = 1.95''$; $s_5 s_6 = 2.13''$. Rays are then drawn to the points s as in the preceding instance, enabling the deflection polygon $C, C_,'' d'_1, d'_2, d'_3, d'_4, d'_5, d'_6$ to be completed, thus giving the deflection distance $\frac{l_1}{4}$.

As l is less than l_1 the pole distance $B\,C$ for the polygon $E\,C\,F$ must be decreased in order to make the moment ordinates M_a correspondingly greater. The proportional scale is used to accomplish this purpose as in the preceding article. The vertical line $A\ C$ is produced downward to O and on it $C\,M$ is laid off equal to l. The horizontal line $M\,N$ is then drawn. A radius $C\,N$ equal to l_1 is then swung from C until it cuts the horizontal line through M at N. The line $C\,N$ is then prolonged to P. Each of the moment ordinates M_a in the trial polygon $E\,C\,F$ must be increased in the ratio $C\,M$ to $C\,N$. Laying off $C\,g$ equal to the moment ordinate $a_2\,c_2$, then drawing the horizontal line g to p gives $C\,p$ which laid off vertically above c'_2 will give the point a'_2 in the true equilibrium polygon. Similarly $C\,g_1$ is the moment ordinate $a_3\,c_3$ and by drawing a horizontal line through g_1 the resulting distance $C\,p_1$ gives the moment ordinate of the true equilibrium polygon or polygonal frame. This process may be repeated for every side of the true equilibrium polygon, but the latter may be more conveniently constructed by the procedure shown in the preceding article.

$C\,P$ is laid off equal to the trial pole distance $C\,B$; then from P the horizontal line $O\,P$ is drawn, giving the true pole distance $C\,O$. By drawing the line $C\ L$ parallel to $E\,F$ there will be cut off on the load line the part 14–L, the vertical reaction at B, and 1–L the vertical reaction at D. As the true closing line must be horizontal the horizontal line $L\,C'$ equal to $C\,O$ is then drawn, giving the true pole C' from which the rays shown are drawn to the proper points on the load line. Starting from

B or D the sides of the true equilibrium polygon may then be drawn parallel to these rays in the usual manner and as shown on Plate V. The true equilibrium polygon $B\ a^1_0\ a^1_1\ a^1_2\ a^1_3\ \ldots\ a^1_{12}\ a^1_{13}$, D is thus completed.

The true pole distance $L\,C^1$ measures 4.15 inches, hence the constant horizontal component of stress in the sides of the true polygonal frame is

$$H = 4.15 \times 60{,}000 = 249{,}000 \text{ lbs.}$$

The moment intercepts, *i.e.*, the vertical lines included between the centre of each joint and the corresponding side of the true polygonal frame, scaled from the original drawing are shown in the following table. The bending moments in foot-pounds at the various joints of the rib are found by simply multiplying the value of H = 249,000 lbs. by the intercepts in feet as given in the table. The moments so found are given in the third column of the table as clearly indicated:

Joint	Moment Intercept	Moments Ft.-Lbs.	Joint	Moment Intercept.	Moments Ft.-Lbs.
B	.0 ft.	0	b_7	+ 2.8 ft.	+ 697,200
b_0	+ .2 "	+ 49,800	b_8	+ 2.0 "	+ 498,000
b_1	+ .5 "	+ 124,500	b_9	+ 0.1 "	+ 24,900
b_2	+ 2.2 "	+ 547,800	b_{10}	− 1.1 "	− 273,900
b_3	+ 4.2 "	+ 1,045,800	b_{11}	− 1.8 "	− 448,200
b_4	+ 5.2 "	+ 1,294,800	b_{12}	− 1.4 "	− 348,600
b_5	+ 5.3 "	+ 1,319,700	b_{13}	.0 "	0
b_6	+ 4.2 "	+ 1,045,800	D	.0 "	0

The bending moments in the above table must be used in determining the dimensions of the various cross-sections of the rib in design work whether for a single metal as in steel ribs or combined materials as in reinforced concrete ribs. If it is a question of capacity of an existing rib these bending moments must be used in connection with the actual dimensions of such a structure. In either case, the greatest stresses produced by positive or negative bending moments must be combined with the intensities of stress produced by the concurrent direct thrusts at any or all joints.

An examination of the true equilibrium polygon shows that the resultant thrust at any joint, taken in the computations as acting at the centre of the latter, is rarely parallel to the axis of the rib at that joint; but in all ordinary cases, as in the present instance, the direction of the resultant thrust varies so little from the axis of the rib that that variation may be ignored. In cases (which seldom occur) where this variation is too much to be neglected, the component of the resultant thrust normal to the surface of the joint (*i.e.*, parallel to the axis of the rib) must be taken instead of the resultant itself.

The resultant thrusts scaled from the original diagram at 60,000 lbs. per linear inch will be found in the following table as will also the transverse shears at the various joints, *i.e.*, those components of the resultant thrust which are parallel to the imaginary joints or normal to the centre line of the rib:

Joint	Shear	Resultant Thrust	Joint	Shear	Resultant Thrust
b_0	+ 21,500 lbs.	509,000 lbs.	b_7	− 28,000 lbs.	252,000 lbs.
b_1	+ 16,500 "	423,000 "	b_8	− 22,500 "	262,000 "
b_2	+ 30,000 "	362,000 "	b_9	− 20,000 "	280,000 "
b_3	+ 20,000 "	313,000 "	b_{10}	− 8,000 "	302,000 "
b_4	+ 11,000 "	276,000 "	b_{11}	− 1,000 "	334,000 "
b_5	− 10,000 "	253,000 "	b_{12}	+ 22,000 "	372,000 "
b_6	− 30,000 "	247,000 "	b_{13}	+ 24,000 "	433,000 "

The shears and thrusts at the joints b_0 and b_{13} belong to the springing joints *B* and *D*.

The plus and minus signs before the shears indicate that the direction of some shears is opposite to the direction of the others in reference to the joints on which they act, as will be evident on reference to the plate.

The position of the true polygonal frame both in this and in the preceding cases shows the effect of the moving load on the left-hand part of the rib. The true equilibrium polygon rises under that part of the rib carrying the greatest load, but falls below the centre line of the rib in much of that part of the span carrying no live load.

Vertical Reactions at Ends of Span.

The vertical reaction at the left end B of the span is shown by the part L–14 of the load line. That part scales 7.35 inches. The reaction at B, therefore, is:

$$7.35 \times 60{,}000 = 441{,}000 \text{ lbs.}$$

The reaction at D is represented by the part L–1 of the load line, which scales 5.93 inches. Hence the reaction at D is:

$$5.93 \times 60{,}000 = 355{,}700 \text{ lbs.}$$

The sum of the two reactions is 797,700 lbs., the total load on the arch.

Art. 86.—Combination of Results for Partial Loads.

If the preceding graphical constructions are made for partial loads, as in the cases taken, they can be used without further construction for two symmetrical loadings of the same amounts, *i.e.*, when live loads of the same total amounts and intensity cover symmetrical parts of the same span. In such cases, however, it will be necessary to make a construction, first for the dead load or own weight of the structure, and then to find the true equilibrium polygon for the moving load alone. The moment intercepts between the centre line of the rib and the equilibrium polygon for each condition of loading are then to be algebraically added for each joint. The results will evidently be the moment intercepts for the concurrent action of the two loadings. In fact, the true equilibrium polygon for the combined two loadings may be readily constructed by laying off these resultant intercepts, positive or negative, from the centre line of the rib.

If the equilibrium polygon for any partial live loading is found the same polygon may evidently be used for the same amount and distribution of live loading placed on the other symmetrical part of the span. The combination of these two polygons in the manner already described will give the true equilibrium polygon for the two symmetrical live loadings acting concurrently. In a similar manner the equilibrium polygon

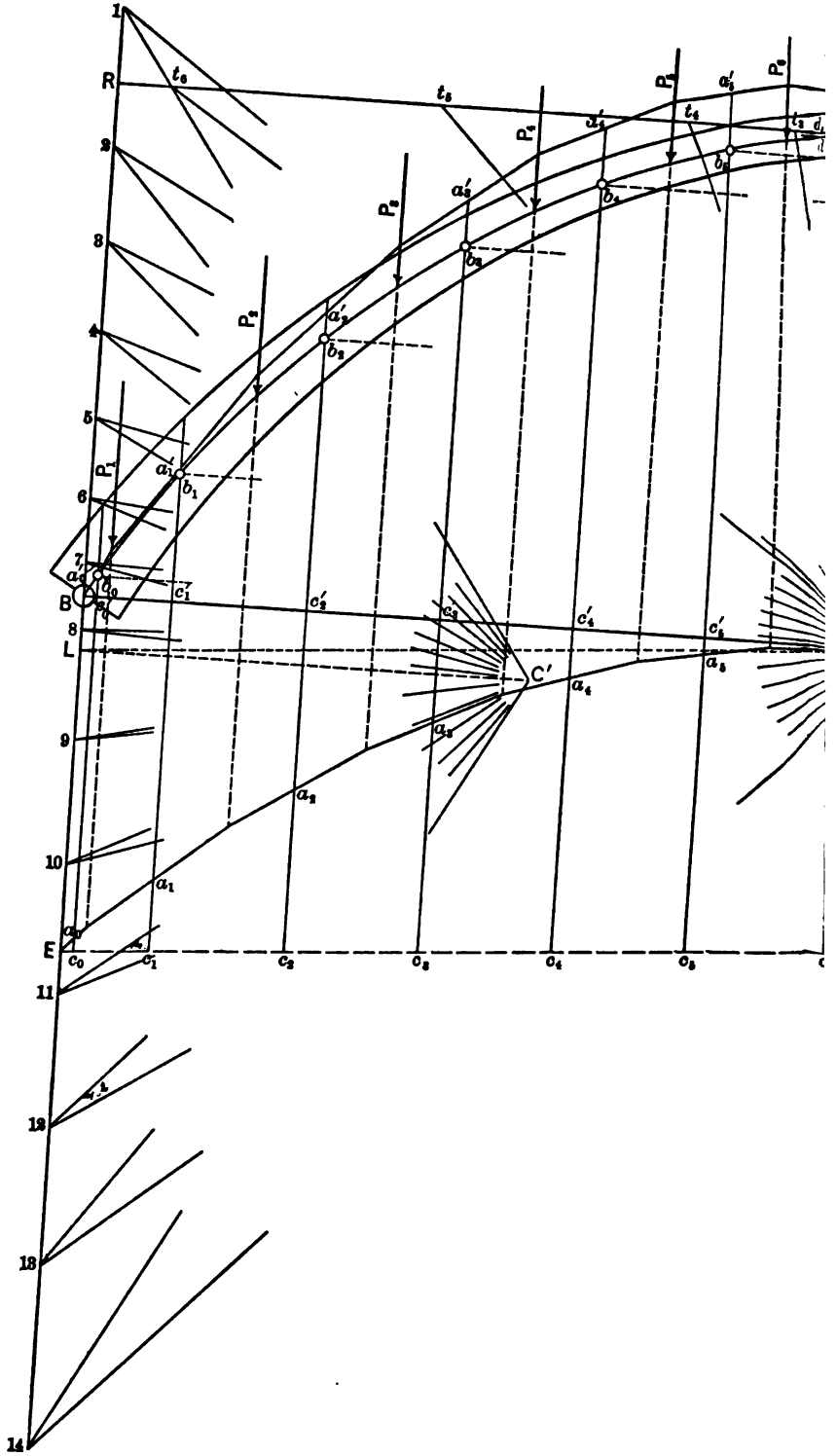

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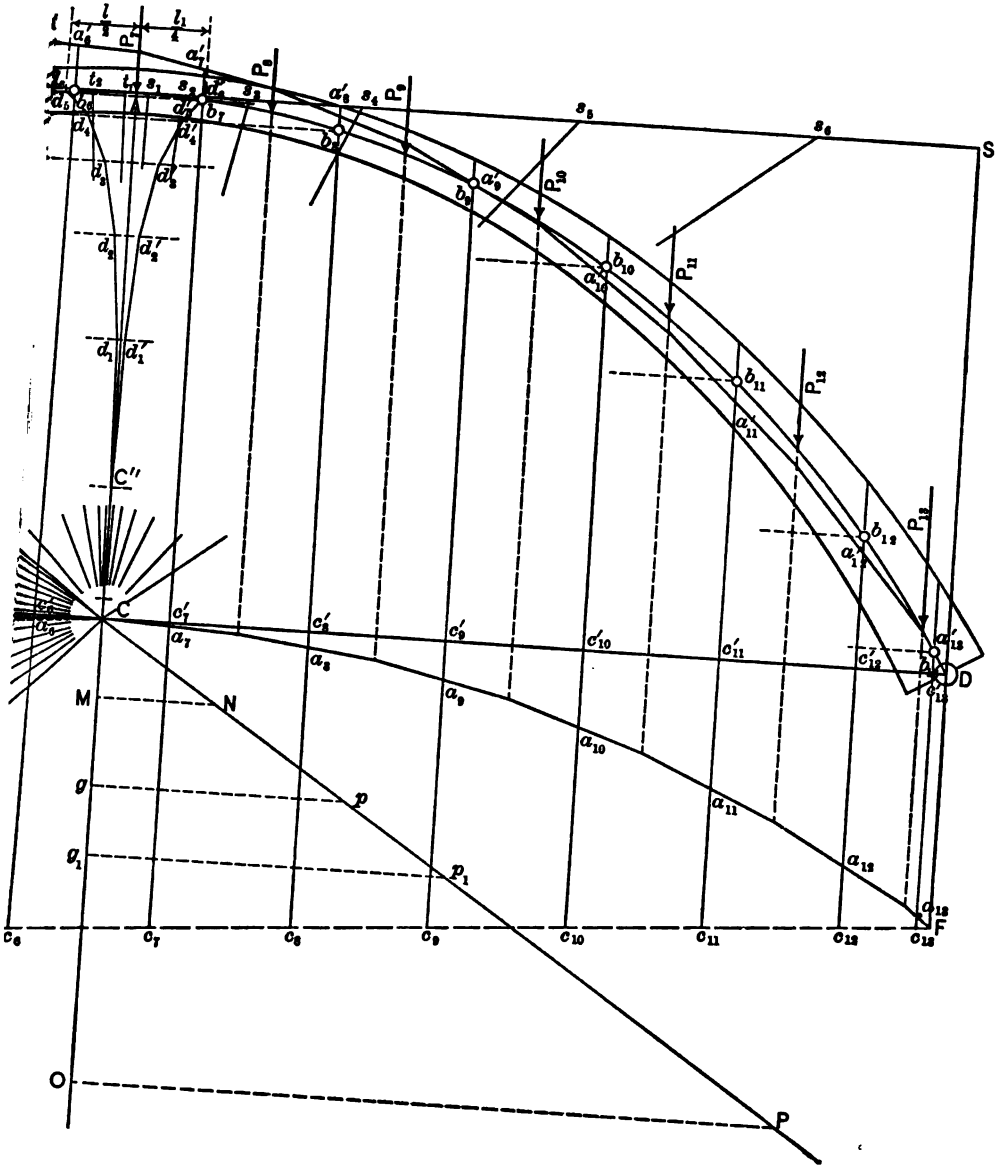
PLATE V.—Art



Rib with Hinged Ends.

so found for the two symmetrical live loadings may be combined with the equilibrium polygon for the dead or fixed loads, so as to give the equilibrium polygon or polygonal frame for the two symmetrical live loadings acting concurrently with the fixed or dead load.

If the live loading should cover half the span, the two equilibrium polygons, one for each half span covered, combined will give the polygonal frame or equilibrium polygon for the live or moving load on the entire span.

In a similar manner the equilibrium polygons for live loading on different parts of the span may be combined, so as to find without further construction the polygonal frame for the simultaneous action of all such live loads.

It is further clear that an equilibrium polygon for any system of loading whatever covering either the whole or partial span may be used as the true equilibrium polygon for any other similar loading of different intensity. In other words, as long as the loadings are similar and are varied by a constant ratio in passing from one system to another the same equilibrium polygon may be made to answer for all. In changing from one to the other it is simply a matter of scale used for the loads.

Art. 87.—Thermal Stresses in the Arch Rib with Ends Fixed.

Thermal stresses are those stresses which are co-existent with any variation of temperature in the structure considered and whose values depend upon that variation. A variation of the temperature of an arch rib will cause a variation in its length, and consequently a deflection at any given point. Although the temperature is supposed to change the ends of the rib must remain in their normal positions, so that the general conditions

$$\sum_D^B \frac{n M}{E I} = 0 \quad \text{and} \quad \sum_D^B \frac{n M y}{E I} = 0 = \sum_D^B \frac{n M x}{E I} \quad \text{hold for}$$

thermal stresses as well as for any other.

Any change of form, such as that arising from the application of loading, will cause corresponding stresses which are to be

determined in precisely the same manner as that used for thermal stresses; in fact, they belong to the same class of stresses.

The coefficient of linear expansion δ of a few structural materials for a variation of temperature of one degree between 32° and 100° Fahr. may safely be taken:

For wrought iron:	$\delta = .000,006,78$
For cast iron:	$\delta = .000,006,25$
For structural steel:	$\delta = .000,006,8$*
For granite:	$\delta = .000,003,81$
For concrete, mortar and rubble masonry:	$\delta = .000,005,55$

Each of these quantities shows the variation of a unit length for a change of one degree Fahrenheit in temperature.

The arch rib considered in the two preceding articles will also be employed here with all the dimensions unchanged. The range of temperature will be taken from 40° above to 40° below the mean. It must be remembered that the range of temperature in the mass of a masonry or reinforced concrete arch rib will not be so great as that of the air, as it requires considerable time for the interior of such a mass to attain the same temperature as the exterior, especially if the cross-section of a rib be large.

If the ends of the rib were free to move horizontally, and if the whole rib were also free to rise or fall in response to changes of temperature, no thermal stresses would exist. If, however, after such free expansion or contraction a sufficient horizontal force be applied along the closing line of the polygonal frame to bring the ends of rib back to their original positions thermal stresses will arise. The deflection upward or downward at the crown corresponding to these thermal stresses is not therefore the whole deflection, but the difference between the whole and that produced by free expansion or contraction. The crown deflection of the rib cannot usually be computed accurately for a given change of temperature, *i.e.*, for a given change of length

* The latest investigations indicate that this value may possibly be taken at .000,006.

of the rib, but it can be found closely enough for stress computations.

If f is the rise of the centre line of the rib at the crown, ΔL the change of length of the latter due to the variation T of the temperature from the mean, and if $n = \frac{f}{l}$, l being the length of the span, Eq. 82 of Art. 35 gives approximately the centre deflection of the rib:

$$\Delta f = \frac{15}{16(5n + 24n^3)} \Delta L \quad . \quad . \quad . \quad . \quad . \quad 19.$$

In this equation:

$$\Delta L = \delta T L \quad . \quad . \quad . \quad . \quad . \quad 20.$$

In this case:

$$n = \frac{47}{151.2} = .311.$$

Also, if $\delta = .000,006,8$; $T° = 40°$; and $L = 186$ feet;

$$\Delta L = .0506 \text{ ft.}$$

Hence:

$$\Delta f = \frac{15}{13.33} .0506 = .057 \text{ ft.}$$

If thermal movement were entirely free the rise or fall of the crown would be:

$$47 \times 40 \times .000,006,8 = .0128 \text{ ft.}$$

Hence the deflection producing thermal stresses is:

$$\Delta^1 f = .057 - .0128 = .0342 \text{ ft.} \quad . \quad . \quad . \quad 21.$$

The difference between the total thermal deflection and that actually required to be used is thus seen to be material for ribs with considerable rise, but the neglect of the difference would be an error on the side of safety.

The effect of the thermal variation is to cause bending moments at the various sections of the rib from which the deflection results. Under such conditions the ends of the rib are kept

in their original position essentially through the action of a horizontal force such as the stress which may be supposed to exist in a horizontal tie. No external forces or loading act upon the rib, except the stress in this imaginary tie. The stresses and bending moments caused by thermal variations are thus seen to be the same as those which would be caused by a horizontal force only having the proper line of action, and the problem resolves itself into finding the proper value and line of action of such a force.

As the ends of the rib are fixed the equations of condition to be fulfilled are Eqs. 5, 6, and 7 of Art. 84, the second member of each of those equations being made equal to zero when the summation extends from one end of the rib to the other. These conditions are obviously identical with those expressed by Eqs. 10, 12, and 14 of the same article, when the rib carries vertical loading. The moments in this case are produced by the action of a single horizontal force.

Eqs. 5 and 6 of Art. 84 adapted to vertical loading in Eqs. 10 and 12 of the same article were used to determine the position of the closing line $H^1 K^1$ of Plate IV. It is clear that if the same equations, *i.e.*, Eqs. 5 and 6, are used in the same manner, they will determine precisely the same closing line $H^1 K^1$ shown on Plate IV. Hence that plate will be used in showing the method for determining temperature stresses in an arch rib with fixed ends.

Inasmuch as $H^1 K^1$ is the true closing line, *i.e.*, the line of action of the horizontal stress or force H produced by change of temperature, the moment ordinates v will be represented by $c^1_0 b_0$, $c^1_1 b_1$, $c^1_2 b_2$, and the remaining normals drawn from each joint of the rib to the closing line $H^1 K^1$. Those measured below the horizontal line $H^1 K^1$ will be considered negative and those measured above the same line positive, as has been done in the preceding articles.

Eqs. 6 and 7 of Art. 84 give the vertical and horizontal deflections of the rib at any point when the summations are taken between either end of the rib and the point where the deflection is desired.

Inasmuch as the rib is symmetrical in reference to a vertical line through the crown the horizontal thermal deflection of that point will be zero, however much the temperature may be changed. Hence Eq. 6 of Art. 84 will be used in connection with the vertical deflection produced by thermal stresses shown by Eq. 21. The thermal bending moment at any point of the rib will be $M = H v$. The following equation, therefore, will give the value of the horizontal stress H which is being sought, the summation being taken between the end of the span and the centre or crown of the rib.

$$\frac{1}{E}\sum_A^B \frac{M x \Delta L}{I} = \frac{H}{E}\sum_A^B \frac{v x \Delta L}{I} = \Delta^1 f = .0342 \text{ ft.} \quad . \quad 22.$$

$$\therefore H = \frac{E}{\sum_A^B \frac{v x \Delta L}{I}} .0342 \quad . \quad . \quad . \quad . \quad 23.$$

By carefully scaling v on the original drawing, the values found in the following table were determined. The other quantities are taken from the data already used in the preceding articles. As the summation extends over half the rib only the numerical values in the tabulation belong to half the span:

x	$\frac{\Delta L}{I}$	v	$\frac{v x \Delta L}{I}$	
1.05 ft.	.03	− 38 ft.	− 1.197	
8 "	.19	− 28.4 "	− 43.168	− 183.235
20.4 "	.25	− 15.0 "	− 76.5	
32.4 "	.35	− 5.5 "	− 62.37	
44.8 "	.49	+ 1.2 "	+ 26.342	
57.2 "	.87	+ 5 3 "	+ 263.749	+ 735.026
69.6 "	.91	+ 7.25 "	+ 444.935	
				+ 551.791

The product $E I$ in Eq. 23 and in the preceding equations represents in the case of a reinforced concrete rib the quantity $E(I + e I_2)$. As is usual in a combined material, if E is the

modulus of elasticity for concrete and E_2 the modulus for steel, the ordinary expression $E\,I$ must be replaced by:

$$E\,I + E_2\,I_2 = E\,(I + e\,I_2) \quad . \quad . \quad . \quad . \quad 24.$$

In this equation e is obviously the ratio found by dividing the modulus of elasticity of steel by the modulus of elasticity of concrete, and it is usually taken as .15 for the purposes of design, although 12 is sometimes used. Inasmuch as the stresses arising from changes of temperature, as well as those arising from changes of shape (to be treated in a later article), take place only after the concrete has attained considerable age, thus giving it full permanent value, e will be taken as 12 for this special purpose, making E = 2,500,000 per square inch.

In all the preceding analysis of this chapter wherever I has been computed for reinforced concrete its numerical value has been found as $I + e\,I_2$.

Introducing in Eq. 23 the numerical quantities shown in the tabulation and taking 2,500,000 per square inch, or 144 × 2,500,000 in this case, as all dimensions are in feet, the horizontal stress H is found to be:

$$H = \frac{144 \times 2{,}500{,}000 \times .0342}{551{,}791} = 22{,}310 \text{ lbs.}$$

The bending moments at the various joints of the rib caused by this horizontal thermal stress H are as follows:

Joint	Moment Ordinate y.	Moment
b_0	− 38 ft.	− 848,670 ft.-lbs.
b_1	− 28.4 "	− 634,250 " "
b_2	− 15.0 "	− 335,000 " "
b_3	− 5.5 "	− 122,830 " "
b_4	+ 1.2 "	+ 26,750 " "
b_5	+ 5.3 "	+ 118,370 " "
b_6	+ 7.25 "	+ 161,920 " "

The bending stresses produced by these moments are to be

combined with other concurrent stresses so as to determine the greatest possible intensities.

It will be observed that the moments at and near the springing joints are large. They show the bending moments required to fix the ends of the rib, on the assumption that the abutment supports remain completely immovable. In reality that condition is impossible. The abutment supports move sensibly within the elastic limit of the materials of which they are composed, and in some cases that movement is easily perceptible. There is, therefore, a material though indeterminate departure from the condition of perfect fixedness assumed which reduces correspondingly the apparent high bending moments at and in the vicinity of the springing joints.

The normal stresses of tension or compression and the shearing stresses caused by thermal changes in each of the imaginary joints of the rib may at once be written after having determined the value of the horizontal stress H given on page 306. If the normal stress on each such joint having the inclination ϕ with a horizontal plane is called N, and if S represents the shearing stress on such a joint, those quantities will have the following values:

$$N = H \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 25.$$

$$S = H \cos \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 26.$$

In other words, the horizontal force H is resolved for each joint into two components, one normal and the other parallel to that joint.

It must be remembered that the normal force N will be compression for a rise in temperature and tension in the case of decreasing temperature. This distinction must be carefully made in determining the resultant intensity of stress in any joint.

The first of the following tabulations gives the *sines* and *cosines* for the various joints in half the rib only, as the two halves are in all respects symmetrical. The second tabulation gives the values of the normal and shearing stresses N and S as indicated by the two preceding equations.

		N	*S*
sin 32° = .53	*cos* 32° = .848	11,820 lbs.	18,910 lbs.
" 38° 35′ = .624	" 38° 35′ = .782	13,920 "	17,440 "
" 48° 15′ = .746	" 48° 15′ = .666	16,640 "	14,870 "
" 56° 45′ = .836	" 56° 45′ = .548	18,640 "	12,220 "
" 66° 45′ = .919	" 66° 45′ = .395	20,490 "	8,810 "
" 76° 15′ = .971	" 76° 15′ = .238	21,650 "	5,310 "
" 85° 5′ = .996	" 85° 5″ = .086	22,210 "	1,920 "

The preceding moments and stresses show all the effects due to thermal changes in the rib. They are to be combined with the moments and stresses caused by the vertical loading and with those due to change of shape of the rib wherever the latter are of sufficient magnitude to make it necessary or advisable to recognize them. As it is quite possible, if not probable, that extreme temperatures may exist with the rib heavily loaded, thermal stresses should be combined with the greatest stresses which may be produced by the moving load.

Art. 88.—Stresses in the Arch Rib with Ends Fixed Due to Change of Shape.

The direct thrust, *i.e.*, compression, in an arch rib produces a change of shape the measure of which is usually taken as the vertical deflection at the crown, although the vertical deflection at any other point might be taken.

It is obviously the principal purpose to find these secondary stresses due to change of shape and to thermal changes so as to make their combination with those due to vertical loading result in the greatest possible intensities of stress at some or all of the joints. The loading to be taken in order to find such stresses due to change of shape of the rib is to some extent a matter of judgment, but it may be stated that usually moving load covering either approximately half the span or the whole of the span should be used. Inasmuch as the graphical constructions of Plate IV have been made for the moving load covering a little less than half the span stresses for the change of shape under the same loading will be found. Precisely the same procedures will be required for moving load over any other part or the whole of the span, or for own weight only.

The stress diagram and equilibrium polygon of Plate IV disclose at once the direct or axial thrust, *i.e.*, compressive stress for all joints throughout the rib. These axial compressive stresses shorten the rib by an amount which can be very simply found, and from that shortening the vertical crown deflection at once results. As was the case with the thermal stresses the full vertical crown deflection is not to be used, since no stresses due to change of shape would be developed if the ends of the rib were free to move horizontally toward each other as the rib is shortened. The amount to be deducted from the full deflection will be clearly shown.

The stress effects due to change of shape, like the effects resulting from thermal changes, may be considered equivalent to those produced by a single horizontal force H with a proper line of action. Inasmuch as the general conditions to be fulfilled by this horizontal force and its bending moments are precisely the same as those fully discussed in connection with the vertical loads and the thermal stresses in Art. 87, the line $H^1 K^1$ of Plate IV will be the line of action of the horizontal force H resulting from change of shape due to axial compression. In other words, the problem is again to find the value of H and then to determine the bending moments and the stresses due to its action.

When the vertical crown deflection is downward the action of the horizontal force H is outward or away from a vertical line through that crown, whereas when that deflection is upward the direction of H will evidently be inward or toward the same vertical line through the crown.

Crown Deflection Due to Direct Thrust.

The direct thrust in each section ΔL of the rib causes a shortening which produces a vertical downward deflection at the crown of the rib in addition to that of bending. The shortening of each ΔL of the rib is to be found throughout its entire length, and the sum of such shortenings is first to be taken. That component of the resultant compressive stress in each side

of the polygonal frame which is parallel to the axis of the rib must be found, and if it be called C, the shortening of each ΔL will be:

$$\frac{C \Delta L}{A E^1} = \frac{C \Delta L}{A E (r + e (1 - r))} \quad . \quad . \quad . \quad . \quad 27.$$

In this expression r is the ratio of the area of concrete section divided by the total area A, so that $(1 - r) A$ is the area of the steel section. As already used e is the ratio of the two moduli of elasticity and it is to be taken as 12 for this purpose. The quantity $E^1 = E (r + e (1 - r))$ is obviously the composite modulus of elasticity for the reinforced concrete section, E being the modulus for concrete and taken here at 2,500,000 per sq. in. Since in this case the steel reinforcement is taken at about $\frac{1}{2}$ per cent. of the total section:

$$r = .995 \text{ and } E^1 = 1.055 \; E = 2{,}637{,}500 \text{ per sq. in.} \quad . \quad 28.$$

The following table shows values of C scaled from the load or force diagram of Plate IV as well as other data needed to compute the desired shortening.

Joint	Axial Thrust C	A	ΔL	$\frac{C \Delta L}{A}$
b_0	528,000 lbs.	24 sq. ft.	4 ft.	160,000
b_1	447,000 "	22.8 " "	19.9 "	716,733
b_2	385,000 "	19.2 " "	16.5 "	624,784
b_3	337,000 "	16.6 " "	14.6 "	573,760
b_4	300,000 "	14.3 " "	13.3 "	553,350
b_5	278,000 "	11.7 " "	12.7 "	715,410
b_6	270,000 "	11.4 " "	12.3 "	585,897
b_7	273,000 "	11.4 " "		
b_8	281,000 "	11.7 " "		3,929,934
b_9	295,000 "	14.3 " "		
b_{10}	315,000 "	16.6 " "		
b_{11}	342,000 "	19.2 " "		
b_{12}	374,000 "	22.8 " "		
b_{13}	432,000 "	24.0 " "		

The values of ΔL and $\frac{C \Delta L}{A}$ are given for half the span

only because both A and ΔL are equal in symmetrical pairs. Hence symmetrical pairs of thrusts C are added and multiplied by the corresponding value of $\frac{\Delta L}{A}$.

The total shortening of the rib will then be, as all dimensions are in feet:

$$\frac{3{,}929{,}934}{144 \times 2{,}502{,}500} = .011 \text{ ft.} = .132 \text{ in.} \quad . \quad . \quad 29.$$

By Eq. 19 of Art. 87, since $n = .311$, there will result:

$$\text{Crown deflection} = \frac{15}{13.33} \times .011 = .01242 \text{ ft.} \quad . \quad 30.$$

If, however, the ends of the rib were free to move horizontally as the shortening under compressive stress takes place, the span would be slightly shortened and the crown would sink, as the length of the rib is 186 feet, by the amount:

$$\frac{.011}{186} \times 47 = .00278 \text{ ft.} \quad . \quad . \quad . \quad . \quad . \quad 31.$$

If a horizontal force H acting along its proper line of action then be applied to the rib with just sufficient magnitude to bring the ends back to the original span, the state of stress due to this change of shape will be reached.

Hence the vertical downward deflection to be used in finding the stresses due to change of shape caused by the direct thrust will be:

$$.01242 - .00278 = .00964 \text{ ft.} \quad . \quad . \quad . \quad . \quad . \quad 32.$$

This correction for the deflection accompanying free shortening, so to call it, is precisely similar to that made in Art. 87, in connection with thermal stresses. Indeed, after the thermal deflections are computed those due to shortening the rib may be found by simple proportion. By referring to the thermal changes of length in Art. 87, for example, the free deflection due to the direct thrust is:

$$\frac{.011}{.0506} \times .0128 = .00278 \text{ ft. (See Eq. 31)} \quad . \quad . \quad . \quad 33.$$

In the same manner other quantities, as bending moments, normal and tangential stresses on the various joints of the rib, may be computed by simple proportion from the values of corresponding quantities due to thermal changes.

Bending Moments, Normal Stresses, and Shearing Stresses Due to Change of Shape Caused by Direct Stresses.

It has been shown by Eq. 32 that the vertical downward crown deflection to be used in finding the stresses due to change of shape caused by the direct thrust is:

$$.00964 \text{ ft.}$$

Consequently the quantities sought can be obtained as shown in the preceding section of this article by multiplying the corresponding quantities due to thermal changes by the ratio:

$$\frac{.00964}{.0342} = .282 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 34.$$

The value of the horizontal stress H for the change of shape caused by the direct thrust will then be:

$$H = 22{,}310 \times .282 = 6{,}290 \text{ lbs.} \quad . \quad . \quad . \quad 35.$$

The desired bending moments, normal or direct stresses N, and shears S, at the various joints of the rib, will be found by multiplying the corresponding quantities due to thermal changes and found on pages 306 and 307, by the above ratio or coefficient .282. This procedure will give the following tabulation of quantities sought:

Joint	Moments	N	S
b_0	− 239,400 ft. lbs.	3,330 lbs.	5,330 lbs.
b_1	− 179,000 " "	3,920 "	4,910 "
b_2	− 94,500 " "	4,680 "	4,200 "
b_3	− 34,700 " "	5,250 "	3,440 "
b_4	+ 7,560 " "	5,780 "	2,480 "
b_5	+ 33,300 " "	6,020 "	1,500 "
b_6	+ 45,700 " "	6,260 "	540 "

As the vertical crown deflection in this case is downward, the direct stresses N will be tensile and must obviously be considered of opposite sign to the compressive stresses with which they may be ultimately combined in order to determine resultant intensities.

These computations complete all required for the treatment of the arch rib with ends fixed. It is seen that, with the exception of the bending moments due to the change of shape caused by the moments produced by vertical loading, the moments and stresses resulting from change of shape are relatively small and may be ignored in many cases of short spans.

Art. 89.—Crown Deflection of Arch Rib with Fixed Ends Due to Vertical Loading.

By Eq. 6 of Art. 84, the vertical downward deflection at the crown A of the rib is:

$$D = \sum_A^B \frac{M x \Delta L}{E I} = \frac{H}{E} \sum_A^B \frac{v x \Delta L}{I} \quad . \quad . \quad 36.$$

The moment intercepts v scaled from Plate IV are given on page 291, where the moments M are also found, while the quantities x, ΔL and I are shown on page 281. The following tabulation exhibits the data required for the solution of the above value of D:

x	$\frac{\Delta L}{I}$	Moments M	$\frac{M x \Delta L}{I}$	
1.05 ft.	.03	− 570,000 ft.-lbs.	− 17,955	
8 "	.19	− 732,000 " "	− 1,112,640	− 3,068,600
20.4 "	.25	− 380,000 " "	− 1,938,000	
32.4 "	.35	0 " "	0	
44.8 "	.49	+ 24,000 " "	+ 526,800	
57.2 "	.87	+ 325,000 " "	+ 16,173,000	+ 17,903,200
69.6 "	.91	+ 19,000 " "	+ 1,203,384	
			Summation =	+ 14,834,600

As the ratio e of the modulus of elasticity of steel divided

by that of concrete is to be taken at 12, and as all dimensions are taken in feet:

$$E = 144 \times 2{,}500{,}000 = 360{,}000{,}000. \quad . \quad . \quad 37.$$

Hence the vertical deflection at the crown of the rib is:

$$D = \frac{14{,}834{,}600}{360{,}000{,}000} = .0412 \text{ ft} = .495 \text{ in.} \quad . \quad . \quad 38.$$

The deflection at any other point than the crown may readily be found by precisely the same general procedure.

It may also be desirable in many cases to determine in the same manner the crown or other deflections separately for the moving load and own weight.

Art. 90.—Stresses and Moments Caused by Thermal Changes in an Arch Rib with Hinged Ends.

In treating thermal stresses in an arch rib with hinged ends the same change of temperature, 40° Fahr., from the mean will be assumed as in a preceding article for a rib with ends fixed. All data, including dimensions of the rib, will be the same as already used for the hinged end arch rib under vertical loading.

The vertical upward or downward crown deflection .0342 ft., given by Eq. 21 of Art. 87, will therefore be used in this case. Obviously the crown deflection for a given change of length of rib will not be precisely the same for ends fixed and hinged, as the character of the curve will be slightly different. As this slight difference in deflection due to small changes in the shape of the curve cannot be simply computed and is too small to be material it will be ignored.

As the ends of the rib are hinged, the line of action of the horizontal stress H produced by the change of temperature will pass through the centres of the two hinges. The thermal bending moments in the rib will therefore be found by simply multiplying that horizontal stress by the vertical ordinates of the various joints of the rib, the origin of the co-ordinates being the centre of the hinge B at the left end of the rib, Plate V, as heretofore.

The value of the horizontal stress H may, therefore, be at once found by Eq. 23 of Art. 87, remembering, however, that the moment ordinate v in that equation is here identical with the ordinate y of the centre line of the rib. The values of v then are to be taken from the tabulated values of y'_0, y_1, etc., on page 282, the value of y'_0 being 1.55 ft., and y_6, *i.e.*, at the joint nearest the crown 46.8 ft.

The following tabulation gives the numerical values of all quantities required to determine the horizontal stress H:

x	$\frac{\Delta L}{I}$	v.	$\frac{v x \Delta L}{I}$
1.05 ft.	.03	+ 1.55 ft.	+ .049
8 "	.19	+ 11.8 "	+ 17.936
20.4 "	.25	+ 24.7 "	+ 125.97
32.4 "	.35	+ 34.1 "	+ 386.694
44.8 "	.49	+ 40.8 "	+ 904.781
57.2 "	.87	+ 44.8 "	+ 2,229.427
69.6 "	.91	+ 46.8 "	+ 2,964.125
		Total	= + 6,628.982

Remembering that as all dimensions are given in feet, the value of the modulus of elasticity E will be taken as $144 \times 2{,}500{,}000$, the value of H found by the equation cited above will be:

$$H = \frac{144 \times 2{,}500{,}000 \times .0342}{6{,}628{,}982} = 1{,}858 \text{ lbs.}$$

The value of this horizontal stress is thus seen to be a little more than 8% of the value found for the rib with fixed ends. If H be multiplied by the ordinates of the various joints, the thermal bending moments at those joints will at once result, and the bending moments so found are given in the tabulation below. Again, if H be resolved into components normal and parallel to the successive joints in half of the rib as indicated by Eqs. 25 and 26 of Art. 87, and as was done for the fixed end rib in a preceding article, the normal thrusts N and the shearing stresses S on each of the joints will be found. The tabulation

given below shows the values of the bending moments $H v$ and the values of N and S for all joints, one-half of the rib being precisely like the other half, as the thermal effects are necessarily symmetrical.

Joint	Moment = Hv.	N	S
b_0	+ 2,880 ft.-lbs.	985 lbs.	1,580 lbs.
b_1	+ 21,920 " "	1,160 "	1,450 "
b_2	+ 45,890 " "	1,387 "	1,240 "
b_3	+ 63,190 " "	1,553 "	1,020 "
b_4	+ 75,810 " "	1,708 "	730 "
b_5	+ 83,240 " "	1,804 "	440 "
b_6	+ 86,950 " "	1,851 "	160 "

These computations complete all required to determine the thermal effects for a given change of temperature. It should be observed that the change of temperature may be either above or below the mean. If there is a rise of temperature above the mean, the horizontal stress H will act from each end of the rib inward, or toward the midpoint of the span, making the normal thrusts N compression, whereas if the temperature is below the mean, causing a downward deflection at the crown, the horizontal stress H will act outward at each end of the rib, making the normal stress N tension. Obviously the shear S will have opposite directions in the two cases.

Art. 91.—Stresses in an Arch Rib with Hinged Ends Due to Change of Shape Caused by Axial Thrust.

The axial thrust along the rib produced by vertical loading will shorten each section of the rib between two consecutive joints, and the total shortening will be the sum of the compressions of all such parts of the rib. It will first be necessary to find that shortening. The effects due to the change of shape produced by the axial thrust will then be precisely like those resulting from the same shortening due to a corresponding change of temperature. The general procedures to be followed are therefore the same as those employed to find the effects of thermal changes,

including the correction in the resulting deflection found by first assuming free contraction, *i.e.*, contraction accompanied by the corresponding shortening of span.

By careful scaling from the original drawing from which Plate V was made, the axial thrusts C along the rib are found to be those shown in the tabulation below. The areas of normal section A and lengths of section of rib ΔL are the same as those used in preceding articles. The shortening of each section ΔL of the rib produced by the axial thrust C has already been given by Eq. 27 of Art. 88, the quantity E^1 having the value shown in connection with that equation. As the quantities ΔL and A are each the same for the two halves of the span, the thrusts C are added for each symmetrical pair of joints and multiplied by ΔL, so that the column headed $\frac{C \Delta L}{A}$ requires values for the half span only, although the results given are for the entire rib.

Joint	Axial Thrust C	A	ΔL	$\frac{C \Delta L}{A}$
b_0	509,000 lbs.	24.0 sq. ft.	4.0 ft.	157,000
b_1	420,000 "	22.8 " "	19.9 "	690,385
b_2	360,000 "	19.2 " "	16.5 "	596,354
b_3	311,000 "	16.6 " "	14.6 "	538,254
b_4	275,000 "	14.3 " "	13.3 "	514,345
b_5	252,000 "	11.7 " "	12.7 "	556,862
b_6	246,000 "	11.4 " "	12.3 "	536,263
b_7	251,000 "	11.4 " "		
b_8	261,000 "	11.7 " "		3,589,463
b_9	278,000 "	14.3 " "		
b_{10}	301,000 "	16.6 " "		
b_{11}	334,000 "	19.2 " "		
b_{12}	371,000 "	22.8 " "		
b_{13}	433,000 "	24.0 " "		

By using the numerical quantity in the preceding tabulation there is at once found:

$$\textit{Shortening} = \frac{3{,}589{,}463}{144 \times 2{,}502{,}500} = .00996 \text{ ft.} = .12 \text{ in.}$$

Again, by using Eq. 82 of Art. 35, as was done before, there will result:

$$\text{Crown deflection} = \frac{15}{13.33} \times .00996 = .0112 \text{ ft.}$$

If, however, there had been free contraction of the rib, *i.e.*, if there had been a corresponding shortening of the span, the crown of the rib would have been depressed, as the rise of the centre line of the rib is 47 feet, by the amount:

$$\frac{.00996}{186} \times 47 = .00252 \text{ ft.}$$

The crown deflection therefore to be used for stresses and moments due to axial thrust along the rib is:

$$.0112 - .00252 = .0087 \text{ ft.}$$

It has already been observed that in cases of thermal changes or effects, or stress effects due to the change of length of rib produced by axial stresses, the values of the horizontal stress H, and hence the bending moments and the values of the direct thrusts and shears, *i.e.*, N and S, will be in direct proportion to the crown deflection upward or downward as the case may be.

Hence in this case the value of the horizontal stress H due to the shortening of the rib by axial thrust may be at once found from the value due to thermal change given in the preceding article where the crown deflection producing stress was shown to be .0342 ft. The value of H sought in this case will then be:

$$H = 1{,}858 \times \frac{.0087}{.0342} = 472 \text{ lbs.}$$

As the ratio $.0087 \div .0342 = .254$, if the moment $= Hv$ and the values N and S of the preceding article be multiplied by .254, the corresponding quantities required in the present case

will at once result. The tabulation given below represents the results of such a procedure:

Joint	Moment = Hv.	N	S
b_0	+ 731 ft. lbs.	250 lbs.	401 lbs.
b_1	+ 5,577 " "	295 "	368 "
b_2	+ 11,756 " "	352 "	315 "
b_3	+ 16,049 " "	394 "	259 "
b_4	+ 19,255 " "	434 "	186 "
b_5	+ 21,143 " "	458 "	112 "
b_6	+ 22,085 " "	470 "	41 "

These complete all the computations of stresses required for the hinged end arch for vertical loads, thermal changes and stresses due to change of shape produced by axial thrust. The normal stresses N are tensile, as the crown deflection if downward.

The preceding results show that the stresses and moments, due to change of shape, produced by axial thrust, are much smaller than those due to thermal changes. Indeed they are so small that in the present case they can be safely ignored.

As is to be expected, all the secondary stresses and moments, *i.e.*, those due to changes of temperature and change of shape due to axial thrust, are much smaller for the rib with hinged ends than for ribs with fixed ends.

Art. 92.—Crown Deflection of Arch Rib with Hinged Ends Carrying Vertical Loading.

It is frequently desirable to determine the crown deflection of a rib with hinged ends under own weight and moving load, the latter being of any given amount. To illustrate the requisite procedure, the deflection will be found for the hinged end arch with its own weight and moving load as taken in Art. 85. The procedure for any amount of moving load by itself or for own weight by itself, or for the latter combined with any arbitrary amount of moving load, will then be obvious without further explanation. All the data to be used are given in Art. 85.

As the moving load covers a little less than half the span, it is clear that the shape of the rib while deflected under the

assumed loading will not be symmetrical in reference to a vertical line through the crown. This condition must be carefully observed in computing the crown deflection desired.

By referring to Art. 83, the general formula for the crown deflection in this case will be found to be:

$$D = \sum_{A}^{B} \frac{M x \Delta L}{E I} \quad . \quad . \quad . \quad . \quad . \quad 39.$$

The values of the quantity $\frac{\Delta L}{I}$ have been used a number of times in the preceding articles, as in Art. 90. The values of the moments M, due to own weight and moving load, are found on page 298. As heretofore, E will be taken at $144 \times 2{,}500{,}000$, because all dimensions are given in feet.

If the centre line of the rib at the crown A, Plate V, be assumed fixed in position, while each half of the rib is subjected to the bending moments due to the vertical loading assumed, Eq. 39, if applied to each half of the rib, will give the vertical deflection of the ends of the rib B and A in respect to the crown. Those deflections will be unequal, but they will both be vertically upward. If then the arch rib in the deflected condition described be supposed to drop so that the two ends B and D assume their normal and proper positions, the crown deflection will be found by combining the two movements, *i.e.*, first, that due to the bending moments in each half of the rib, and second that due to the dropping of the rib to its two points of support B and D.

The first tabulation below gives the numerical quantities from the articles already indicated to be used in finding the vertical deflection for each half of the rib, assuming the centre line of the rib at the crown to be fixed in position.

x	$\frac{\Delta L}{I}$	Moments $B - A$ Ft.-Lbs.	Moments $D - A$ Ft.-Lbs.
1.05 ft.	.03	+ 49,800	0
8.0 "	.19	+ 124,500	− 348,600
20.4 "	.25	+ 547,800	− 448,200
32.4 "	.35	+ 1,045,800	− 273,900
44.8 "	.49	+ 1,294,800	+ 24,900
57.2 "	.87	+ 1,319,700	+ 498,000
69.6 "	.91	+ 1,045,800	+ 697,200

In this tabulation it should be carefully observed that for the moments $D-A$ the co-ordinate x is measured from D.

By substituting these numerical quantities in Eq. 39 the results given below are at once found:

$\frac{B-A}{\frac{M x \Delta L}{I}}$	$\frac{D-A}{\frac{M x \Delta L}{I}}$
+ 1,569	0
+ 189,240	− 529,872
+ 2,793,780	− 2,284,920
+ 11,859,372	− 3,106,026
+ 28,423,450	+ 547,095
+ 66,543,551	+ 24,782,472
+ 66,579,393	+ 44,386,262
+ 176,390,346	+ 63,795,011

Hence, the upward deflection of the end of the rib B in reference to the crown will be:

$$\frac{176,390,346}{144 \times 2,500,000} = .49 \text{ ft.}$$

Similarly the upward deflection of the right-hand end D of the rib in reference to the crown will be:

$$\frac{63,795,011}{144 \times 2,500,000} = .177 \text{ ft.}$$

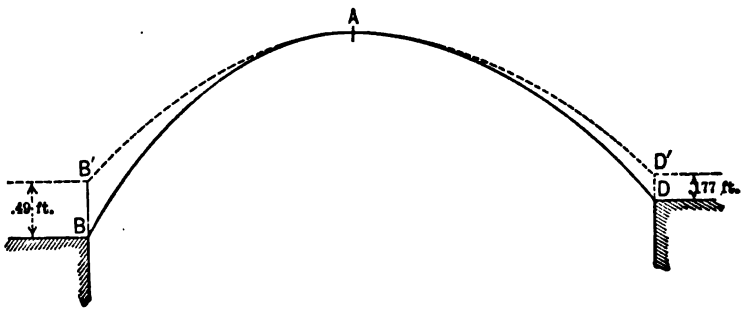


FIG. 6.

The rib will then be in the position indicated by the broken line in Fig. 6, the point B having been raised vertically to B^1

and D to D^1. If the rib now be supposed to drop to its original position so that B^1 may be brought back to B and D^1 to D, the crown A will be deflected by the amount:

$$\frac{.49 + .177}{2} = .3335 \text{ ft.} = 4 \text{ in.} \quad . \quad . \quad . \quad . \quad 40.$$

The actual vertical deflection of the crown under the loading taken will therefore be one-third of a foot, or 4 inches.

The axis of the rib at the crown A will, however, no longer be horizontal but inclined upward in passing from the left of the crown to the right, since $B^1 B$ is greater than $D^1 D$. The tangent of inclination of the axis of the rib at the crown A will then be:

$$\frac{.49 - .177}{151.2} = .00207$$

The deflection and change of slope at any other point of the rib may be determined in a precisely similar manner, the summations being taken from the ends of the rib to such points as may be desired. Obviously the same general procedure will be followed for any other amount of moving load than that taken, also, either for own weight alone or for moving load alone.

CHAPTER VIII

The General Analytic Theory of Elastic Arch Ribs According to the Law of Least Work.

Art. 93.—General Considerations on the Theory of Arch Ribs.

An arch rib differs from an ordinary bridge truss chiefly in that the reactions at the two ends of the rib are not vertical, a horizontal component always existing. The arch rib, like the truss, may be continuous, but that continuity may be such as to join the ends of the rib to the piers only which support them. The ends of the arch rib may also be hinged or jointed, and again there may be a hinge or joint at the crown. If the hinge at the crown is absent the arch rib is statically indeterminate, whether its ends are hinged or continuous, but if the rib has hinges both at the ends and at the crown it is statically determinate, requiring only the ordinary statical equations of equilibrium for the complete analysis of its stresses.

In the cases of statical indetermination, *i.e.*, when the ends are continuous or, when there are hinges at the ends only, it is necessary to employ equations of condition which depend upon the elastic properties of the materials used in the rib, in addition to the ordinary statical equations of equilibrium. These equations may be based upon others giving deflections by the common theory of flexure, as in the case of stiffened suspension bridges, or they may be found by the aid of the law or principle of least work, also as illustrated in the theory of suspension bridges. In fact, an arch rib is essentially an inverted, stiffened, suspension bridge, by which transformation the stiffened cable becomes the arch rib in compression, the bending moments in the stiffening-truss or stiffening bracing of the suspension bridge corresponding to the bending moments of the arch rib. It is clear, therefore,

that the arch rib must sustain a combination of bending or flexure and direct compression, and its cross-section must be designed accordingly. That cross-section may be solid, like either a plate girder or a plain concrete arch of a single material, or a solid reinforced concrete section; or again, the rib may be an articulated structure with open bracing, as in many steel arch ribs for railway station buildings and other similar structures. It is equally clear that the cross-section of the arch rib may be subjected to transverse shear in a normal cross-section or to a shear in some other direction, depending upon the character of the loading.

Arch ribs may be subjected to loading in any direction, *i.e.*, normal to the exterior surface, as in the case of wind pressure, or in a vertical direction, as in the case of many plain and reinforced concrete arches, or in any other direction. A proper general theory for such structures must therefore provide for loading in any direction and for the effects of bending moments, normal stresses, and shears.

In the following articles will be given a general analysis for any conditions of loading, based upon the principle of least work. The graphical method based upon the use of elastic deflections due to flexure is given in the preceding chapter.

Art. 94.—Arch Rib with Fixed Ends—Stresses and Moments —Total Work Expended.

The general treatment of an arch rib with fixed ends presupposes any curve for the rib and any system of loads, as illustrated in Fig. 1. The loads P_1, P_2, P_3, etc., acting at any points and in any directions, are shown as applied to the centre line of the rib having the span $F K$ or l. The points F and K are supposed to be at the same elevation and F is the origin of the co-ordinates x and y of the curved axis of the rib.

The points c and c^1 represent any normal sections of the rib, with which the lines of action of P_1, P_2, etc., make the respective angles ϕ_1, ϕ_2, etc., the normal sections themselves making the

angles ϕ with a horizontal plane. The lines of action of P_1, P_2, etc., make the angles α_1, α_2, etc., with the horizontal, so that:

$$\alpha_1 + \phi_1 = \alpha_2 + \phi_2 = \text{etc.} = \phi.$$

It is to be carefully observed that α_1, α_2, etc., ϕ_1, ϕ_2, etc., are positive as measured clockwise, *i.e.*, right-handed from a horizontal line or from the lines of action of P_1, P_2, etc. The same observation applies to the angles ϕ locating the normal sections of the rib and measured clockwise from the same horizontal line as α_1, α_2, etc.

It will first be necessary to determine the total normal and tangential (shearing) forces or stresses in each normal section of the rib. The requisite expression may readily be written as follows, H and V being the horizontal and vertical components respectively of the reaction at F, while H^1 and V^1 will be the corresponding components at K.

The component of any load P_1, Fig. 1, acting perpendicular to any normal section c or c^1 of the rib, is $P_1 \sin \phi_1$, and the tangential or shearing component of the same load is $P_1 \cos \phi_1$. Similarly the horizontal and vertical components of the reaction at F will have normal and tangential components, respectively, $H \sin \phi$, $H \cos \phi$, and $V \cos \phi$, $- V \sin \phi$. If the angle made by any force P_1 with the normal section is negative, like $- \phi_3$ for P_3, the normal component becomes $- P_3 \sin \phi_3$, although the tangential component may still remain $+ P_3 \cos \phi_3$.

As the forces and moments to which any part $F\,c$ or $F\,c^1$ of the arch is subjected are being considered, a positive value of the normal force acting on any joint c or c^1 indicates compression and a negative force, tension.

The moment about the joint c or c^1 of every load or force acting on any part $F\,c$ or $F\,c^1$ of the rib must also be written. For this purpose let a_1 and b_1 be the horizontal and vertical co-ordinates of the point of application of any force or load on the axis of the arch, and let x and y be the corresponding co-ordinates of the normal section c or c^1. Then the lever arm of any such force P_1 will be $(y - b_1) \cos \alpha_1 + (x - a_1) \sin \alpha_1$.

Hence the desired moment for P_1 will be:

$$P_1 \{(y - b_1) \cos \alpha_1 + (x - a_1) \sin \alpha_1\} \quad . \quad . \quad . \quad 1.*$$

For H, this expression becomes, since $\alpha_1 = 0$, and $a_1 = b_1 = 0$:

$$H y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 2.$$

And for V, since $\alpha_1 = -90°$ and $a_1 = b_1 = 0$:

$$-V x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 3.$$

As the ends of the arch rib in this case are fixed, each of the end joints F and K is subjected to a bending moment of such magnitude and direction as is necessary to secure fixedness under any loading. In general the moments will not be equal nor of the same sign. Let this external or fixing moment acting at F be represented by M_0. The resultant bending moment at any section of the rib will then be the algebraic sum of M_0 and the

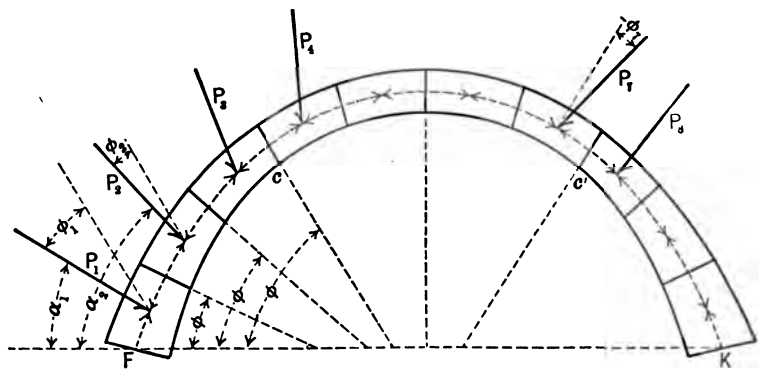


FIG. 1.

ordinary non-continuous moment in the curved beam or rib acted upon by H, V, P_1, P_2, etc., as shown below in Eq. 6.

The following general expressions for the normal and tangential components of the external loads N and S on any normal

*This equation may perhaps be written more simply by remembering that $P_1 \cos \alpha$, and $P_1 \sin \alpha$, are the horizontal and vertical components of P_1 acting with the lever arms $(y - b_1)$ and $(x - a_1)$, respectively.

section of the rib, and for the moments M of those loads or forces about the centre of that normal section may now be written as follows:

$$N = H \sin\phi + V \cos\phi + \sum_0^{\phi} P_1 \sin\phi_1 \quad . \quad . \quad . \quad . \quad . \quad 4.$$

$$S = H \cos\phi - V \sin\phi + \sum_0^{\phi} P_1 \cos\phi_1 \quad . \quad . \quad . \quad . \quad . \quad 5.$$

$$M = M_0 + Hy - Vx + \sum_0^{\phi} P_1 \Big\{ (y - b_1)\cos\alpha_1 + (x - a_1)\sin\alpha_1 \Big\} 6.$$

These equations are obviously perfectly general and they recognize the radial shear S.

In all that follows, there will be used:

$$M^1 = \sum_0^{\phi} P_1 \Big\{ (y - b_1) \cos\alpha_1 + (x - a_1) \sin\alpha_1 \Big\} \quad . \quad 6a.$$

In accordance with the law of least work, H, V, and M_0 must have such values that the total work W expended in stressing all the members of the arch rib will be a minimum. The expression for that total expended work is, if L is the length of the rib measured along its axis:

$$W^* = \frac{1}{2}\int \frac{M^2}{EI}\,dL + \frac{1}{2}\int \frac{N^2}{EA}\,dL + \frac{6}{5}\int \frac{S^2}{GA^1}\,dL =$$

$$\frac{1}{2E}\left(\int \frac{M^2}{I}\,dL + \int \frac{N^2}{A}\,dL\right) + \frac{6}{5G}\int \frac{S^2}{A^1}\,dL \quad . \quad . \quad 7.$$

In this equation, I is the moment of inertia of any normal section of the rib about its neutral axis, while A is the area of that cross-section and A^1 is the normal sectional area on which the radial shear S acts. The area A may or may not be equal to A^1. Each dL is an indefinitely short part of the rib or of the member carrying the shear S. G is the modulus of elasticity for shear, and E that for direct stress.

If the rib is of rectangular cross-section and of plain concrete,

* "Elasticity and Resistance of Materials," by Wm. H. Burr, Art. 29.

b being the breadth of the rib and h the depth $A^1 = b\,h$ and S will be the total radial shear (Eq. 5) at any normal section. If the concrete be reinforced, A^1 will have the same value $b\,h$ plus such allowance as may be made for the steel part of the section. The latter may be taken in some cases as an area of concrete section sufficient in amount to afford shearing resistance equivalent to that of the steel.

If the rib should be of steel-plate girder section, A^1 would probably be taken as the area of the normal section of the web plate, possibly allowing for the rivet holes.

Whenever the shear section A^1 is constant, we may at once write:

$$\frac{6}{5G}\int \frac{S^2}{A^1}\,dL = \frac{6}{5GA^1}\int S^2\,dL \quad . \quad . \quad . \quad 8.$$

If the arch rib is an articulated truss with the usual chords and web members, as in Fig. 2, the expression for the work of the shear S in Eqs. 7 and 8 must again be modified. If there be either a single or a double system of web members, the shear S remains

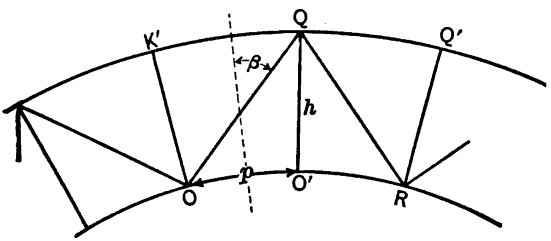


FIG. 2.

constant throughout the panel length $p = O\,O^1$. The tension or compression in the web member $O\,Q$ will then be $S \sec \beta$. The length of the web member $O\,Q$ will be $p \operatorname{cosec} \beta$. Hence if A'' is the area of the normal cross-section of the web member $O\,Q$, the work expended in stressing it will be:

$$W'' = \frac{S^2 \sec^2 \beta \operatorname{cosec} \beta}{2\,E\,A''}\,p \quad . \quad . \quad . \quad . \quad . \quad 9.$$

If there are two systems of bracing, Eq. 9 will be in no way changed, but A'' will then represent the area of the normal cross-section of the two web members found in the panel p.

If the system of web bracing is diagonal and radial, as $O\,Q\,O'\,Q'$, Fig. 2, p in Eq. 9 should be displaced by $h \tan \beta$, after which β is to be made zero. Introducing these changes in Eq. 9, the work expended in stressing the radial web member $O'\,Q$ will be:

$$W''' = \frac{S^2 h}{2 E A'''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 10.$$

Obviously, A''' is the area of normal section of the radial web member.

In the general application of Eq. 7, the third term of the second member of that equation is to be displaced by Eq. 9 for each part of the integration covering a panel p, S being constant for each panel, but varying from panel to panel.

If diagonal and radial bracing is used in the rib the sum of the second members of Eqs. 9 and 10 must displace the third term of the second member of Eq. 7, as explained above, instead of that of Eq. 9 alone.

Art. 95.—Arch Ribs with Fixed Ends—Application of Principle of Least Work.

It is stated in connection with Eq. 7 of the preceding article that H, V and M_0 must have values which will satisfy the law of least work. This requires that the first partial derivatives of W (Eq. 7), in respect to H, V, and M_0 as independent variables, shall each be zero. Hence, displacing the integral sign by the more general summation sign, and making $G = n\,E$, Eq. 7 will give:

$$\frac{\delta W}{\delta H} = \frac{dL}{E}\left\{\sum \frac{M}{I}\frac{\delta M}{\delta H} + \sum \frac{N}{A}\frac{\delta N}{\delta H} + \frac{12}{5n}\sum \frac{S}{A'}\frac{\delta S}{\delta H}\right\} \quad . \quad 11.$$

$$\frac{\delta W}{\delta M_0} = \frac{dL}{E} \sum \frac{M}{I}\frac{\delta M}{\delta M_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 12.$$

$$\frac{\delta W}{\delta V} = \frac{dL}{E}\left\{\sum \frac{M}{I}\frac{\delta M}{\delta V} + \sum \frac{N}{A}\frac{\delta N}{\delta V} + \frac{12}{5n}\sum \frac{S}{A'}\frac{\delta S}{\delta V}\right\} \quad . \quad 13.$$

The quantity n $\left(\text{the ratio } \frac{G}{E}\right)$ is known.

In the actual use of these equations the quantity dL will have finite values each representing the length of a section of the rib, or some other finite length.

Again, the third term in the second members of Eqs. 11 and 13 will be modified for particular cases as already set forth in connection with Eqs. 9 and 10, but without changing the principal elements of those terms.

By differentiating Eqs. 4, 5, and 6:

$$\frac{\delta M}{\delta H} = y \qquad \frac{\delta M}{\delta V} = -x$$

$$\frac{\delta N}{\delta H} = \sin\phi \qquad \frac{\delta N}{\delta V} = \cos\phi$$

$$\frac{\delta S}{\delta H} = \cos\phi \qquad \frac{\delta S}{\delta V} = -\sin\phi$$

$$\frac{\delta M}{\delta M_0} = 1$$

These values are to be inserted in Eqs. 11, 12, and 13. Then writing ΔL for dL and extending the summations over the entire length L of the rib, each of those equations must be placed equal to zero, remembering that E divides out and disappears:

$$\left[\sum \frac{M}{I} y + \sum \frac{N}{A} \sin\phi + \frac{12}{5n} \sum \frac{S}{A'} \cos\phi\right]_0^L \Delta L = 0 \quad . \quad 14.$$

$$\sum_0^L \frac{M}{I} \Delta L = 0 \quad . \quad . \quad . \quad . \quad . \quad 15.$$

$$\left[-\sum \frac{M}{I} x + \sum \frac{N}{A} \cos\phi - \frac{12}{5n} \sum \frac{S}{A'} \sin\phi\right]_0^L \Delta L = 0 \quad . \quad 16.$$

The values of N, S, and M, given in Eqs. 4, 5, and 6, must next be substituted in Eqs. 14, 15, and 16, giving three equations from which H, V, and M_0 may at once be found.

To simplify the forms of those equations let the following notation be used:

$$a = -\frac{\left[\sum \frac{y\,x}{I} - \sum \frac{\sin\phi\cos\phi}{A} + \frac{12}{5n}\sum \frac{\sin\phi\cos\phi}{A'}\right]\Delta L}{\left(\sum \frac{y^2}{I} + \sum \frac{\sin^2\phi}{A} + \frac{12}{5n}\sum \frac{\cos^2\phi}{A'}\right)\Delta L = B} \quad . \quad 17.$$

$$b = \frac{\sum \frac{y}{I}\Delta L}{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 18.$$

$$c = \frac{\left[\sum \frac{M'y}{I} + \sum \frac{\sin\phi}{A}\left(\sum_0^{\phi} P_1 \sin\phi_1\right) + \frac{12}{5n}\sum \frac{\cos\phi}{A'}\left(\sum_0^{\phi} P_1 \cos\phi_1\right)\right]\Delta L}{B} \quad . \quad . \quad . \quad 19.$$

$$a' = -\frac{\sum \frac{x}{I}\Delta L}{\sum \frac{y}{I}\Delta L};\quad b' = \frac{\sum \frac{\Delta L}{I}}{\sum \frac{y}{I}\Delta L};\quad c' = \frac{\sum \frac{M'}{I}\Delta L}{\sum \frac{y}{I}\Delta L} \quad . \quad . \quad 20.$$

$$a'' = -\frac{\left[\sum \frac{x^2}{I} + \sum \frac{\cos^2\phi}{A} + \frac{12}{5n}\sum \frac{\sin^2\phi}{A'}\right]\Delta L}{\left(\sum \frac{x\,y}{I} - \sum \frac{\sin\phi\cos\phi}{A} + \frac{12}{5n}\sum \frac{\sin\phi\cos\phi}{A'}\right)\Delta L = C} \quad . \quad 21.$$

$$b'' = \frac{\sum \frac{x}{I}\Delta L}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 22.$$

$$c'' = \frac{\left[\sum \frac{M'x}{I} - \sum \frac{\cos\phi}{A}\left(\sum^{\phi} P_1 \sin\phi_1\right) + \frac{12}{5n}\sum \frac{\sin\phi}{A'}\left(\sum_0^{\phi} P_1 \cos\phi_1\right)\right]\Delta L}{C} \quad . \quad . \quad 23.$$

The values of B and C are obvious, as shown, and M^1 is given by Eq. 6*a*.

When the parts into which the rib may be divided, as indicated in Fig. 1, are all equal, ΔL will divide out and disappear from Eqs. 17, 18, 19, 20, 21, 22, and 23; but when, as in the general case, those parts are unequal, ΔL must be made a factor of each quantity preceded by the summation sign Σ, as shown in Eqs. 18, 20, and 22, for ΔL is then a variable quantity and of finite value in the actual application of the formulæ.

Making the substitutions indicated above, and using the notation shown, the following equations will result:

$$H + a\,V + b\,M_0 + c = 0 \quad . \quad . \quad . \quad . \quad 24.$$

$$H + a'\,V + b'\,M_0 + c' = 0 \quad . \quad . \quad . \quad . \quad 25.$$

$$H + a''\,V + b''\,M_0 + c'' = 0 \quad . \quad . \quad . \quad 26.$$

Subtracting Eq. 24 from Eq. 25, and then subtracting Eq. 25 from Eq. 26:

$$V\,(a' - a) + M_0\,(b' - b) + c' - c = 0 \quad . \quad . \quad . \quad . \quad 27.$$

$$V\,(a'' - a') + M_0\,(b'' - b') + c'' - c' = 0 \quad . \quad . \quad 28.$$

Eliminating V in a similar manner by using Eqs. 27 and 28:

$$M_0 = \frac{(c' - c)\,\dfrac{a'' - a'}{a' - a} - (c'' - c')}{(b'' - b') - (b' - b)\,\dfrac{a'' - a'}{a' - a}} \quad . \quad . \quad . \quad . \quad 29.$$

Eq. 27 will then give:

$$V = M_0\,\frac{b - b'}{a' - a} - \frac{c' - c}{a' - a} \quad . \quad . \quad . \quad . \quad . \quad 30.$$

Or, by Eq. 28:

$$V = M_0\,\frac{b' - b''}{a'' - a'} - \frac{c'' - c'}{a'' - a'} \quad . \quad . \quad . \quad . \quad 31.$$

Then by Eq. 24 (Eq. 25 or Eq. 26 may be similarly used):

$$H = -\,a\,V - b\,M_0 - c \quad . \quad . \quad . \quad . \quad 32.$$

In Eqs. 14 to 23, both inclusive, the summation sign Σ is to be taken as covering the entire rib, *i.e.*, $\sum_0^L$ except in terms where the limits ϕ and o are shown, as in two terms each of Eqs. 19 and 23.

In this latter case, however, the summation

$$\sum \frac{\cos\phi}{A}\left(\sum_0^{\phi} P_1 \sin\phi_1\right)$$

is to be taken between the limits a_1 (the point of application of the force or load nearest to the left end of the span) and l, the right end of the span. In fact, the summation for each load is to extend from its point of application to the right end of the span. This procedure will be illustrated later by actual application.

Care must be taken in applying the preceding equations to determine properly the value of the sectional area A on which the normal stress N acts when the rib is of reinforced concrete or any other combined material. If e is the ratio between the moduli of elasticity for concrete and steel as given in Eq. 46 of Art. 97, and if A_1 and A_2 (as used hereafter) are the sectional areas of concrete and steel, respectively, the quantity to be used for A, the total area of section, is:

$$A_1 + e A_2 = A + (e - 1) A_2 \quad . \quad . \quad . \quad 32a.$$

In the same case the quantity $E I$ must be given the value, as shown by Eqs. 49 and 50 of Art. 97, I_1 and I_2 being the moments of inertia of the concrete and steel sections respectively:

$$E_1 I_1 + E_2 I_2 = E_1 [I + (e - 1) I_2] \quad . \quad . \quad . \quad 32b.$$

These two equations have been used in the numerical applications of this analysis to an actual reinforced concrete arch in Art. 98, I being the moment of inertia of the entire section.

The preceding equations are perfectly general for any loading whatever and for any curve of rib, whether symmetrical or un-

symmetrical. They can be materially simplified for symmetrical curves and vertical loading.

Art. 96.—Arch Ribs with Fixed Ends—Symmetrical Arches.

When the arch curve is symmetrical in reference to a vertical line through the crown, some of the equations, 17 to 23, of the preceding article may be materially simplified.

In this case the joints located by the angle ϕ, Fig. 1, will be symmetrically placed in reference to the crown of the arch, whether their number be even or odd. Then, since *cos* $(90° - \phi) = -$ *cos* $(90° + \phi)$ and *sin* $(90° - \phi) =$ *sin* $(90° + \phi)$, and remembering that for symmetrical ribs, A, A', and ΔL have each the same value at equal distances either side of the crown:

$$\sum_0^l \frac{\sin\phi\cos\phi}{A}\Delta L = 0 = \sum_0^l \frac{\sin\phi\cos\phi}{A'}\Delta L.$$

The summation limits l and o have the same significance as L and o.

This result reduces the numerator of the second member of Eq. 17, and the denominator of the second member of Eq. 21, to the single term, $\sum_0^l \frac{yx}{I}\Delta L$.

Furthermore, this latter summation is composed of pairs of terms of the kind:

$$\left(\frac{yx}{I} + \frac{(l-x)y}{I}\right)\Delta L = l\frac{y}{I}\Delta L;$$

Hence:

$$\sum_0^l \frac{yx}{I}\Delta L = l\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I} \quad . \quad . \quad . \quad . \quad 33.$$

It is also evident, for these conditions of symmetry, that:

$$\sum_0^l \frac{y^2}{I}\Delta L = 2\sum_0^{\frac{l}{2}} \frac{y^2}{I}\Delta L; \sum_0^l \frac{\sin^2\phi}{A}\Delta L = 2\sum_0^{\frac{l}{2}} \frac{\sin^2\phi}{A}\Delta L;$$

and

$$\sum_0^l \frac{\cos^2\phi}{A'}\Delta L = 2\sum_0^{\frac{l}{2}} \frac{\cos^2\phi}{A'}\Delta L \quad . \quad . \quad . \quad 34.$$

In Eqs. 18 and 20, obviously:

$$\sum_0^l \frac{y}{I}\,\Delta L = 2\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I} \quad . \quad . \quad . \quad . \quad . \quad 35.$$

The summation in the numerator of the first of Eq. 20 is composed of pairs of terms:

$$\frac{x\,\Delta L}{I} + \frac{(l-x)\,\Delta L}{I} = l\,\frac{\Delta L}{I};$$

Hence:

$$\sum_0^l \frac{x\,\Delta L}{I} = l\sum_0^{\frac{l}{2}} \frac{\Delta L}{I} \quad . \quad . \quad . \quad . \quad . \quad 36.$$

Finally:

$$\sum_0^l \frac{\Delta L}{I} = 2\sum_0^{\frac{l}{2}} \frac{\Delta L}{I} \quad . \quad . \quad . \quad . \quad 37.$$

Introducing these modifications in Eqs. 17 to 23, including both, there will result:

For Symmetrical Arches with Symmetrical Ribs:

$$a = -\frac{l}{2}\,\frac{\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}}{\left(\sum_0^{\frac{l}{2}} \frac{y^2\,\Delta L}{I} + \sum_0^{\frac{l}{2}} \frac{\sin^2\phi\,\Delta L}{A} + \frac{12}{5n}\sum_0^{\frac{l}{2}} \frac{\cos^2\phi\,\Delta L}{A'}\right) = B_1} \quad . \quad 38.$$

$$b = \frac{\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}}{B_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 39.$$

$$c = \frac{\sum_c^l \frac{M'\,y\,\Delta L}{I} + \sum_0^l \frac{\sin\phi}{A}\left(\sum_0^{\phi} P_1 \sin\phi_1\right)\Delta L + \frac{12}{5n}\sum^l \frac{\cos\phi}{A'}\left(\sum_0^{\phi} P_1 \cos\phi_1\right)\Delta L}{2\,B^1} \quad . \quad . \quad . \quad 40.$$

$$a' = -\frac{l}{2}\frac{\sum_0^{\frac{l}{2}} \frac{\Delta L}{I}}{\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}}; \quad b' = \frac{\sum_0^{\frac{l}{2}} \frac{\Delta L}{I}}{\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}}; \quad c' = \frac{\sum_0^{l} \frac{M'\,\Delta L}{I}}{2\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}} \quad . \quad 41.$$

$$a'' = -\frac{\sum_0^{l} \frac{x^2\,\Delta L}{I} + 2\sum_0^{\frac{l}{2}} \frac{\cos^2\phi\,\Delta L}{A} + \frac{24}{5n}\sum_0^{\frac{l}{2}} \frac{\sin^2\phi\,\Delta L}{A'}}{l\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}} \quad . \quad 42.$$

$$b'' = \frac{\sum_0^{\frac{l}{2}} \frac{\Delta L}{I}}{\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}} = b' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 43.$$

$$c'' = \frac{\sum_0^{l} \frac{M'\,x\,\Delta L}{I} - \sum_0^{l} \frac{\cos\phi}{A}\left(\sum_0^{\phi} P_1 \sin\phi_1\right)\Delta L + \frac{12}{5n}\sum_0^{l} \frac{\sin\phi}{A'}\left(\sum_0^{\phi} P_1 \cos\phi_1\right)\Delta L}{l\sum_0^{\frac{l}{2}} \frac{y\,\Delta L}{I}} \quad . \quad . \quad . \quad 44.$$

These equations are applicable for any loading acting in any direction; it is only required that the points of loading shall be symmetrically placed.

As $b'' = b'$, Eq. 29 gives:

$$M_0 = \frac{\frac{a' - a}{a'' - a'}\left(c'' - c'\right) - c' + c}{b' - b} \quad . \quad . \quad . \quad 44a.$$

Also, by Eq. 31: $$V = \frac{c' - c''}{a'' - a'} \quad . \quad . \quad . \quad . \quad . \quad . \quad 44b.$$

The value of H given by Eq. 32 remains unchanged.

It is to be carefully observed in applying the preceding equations that $\sum_0^{\phi} P_1 \sin \phi_1$ is the total, normal, external force or loading on any joint located by the angle ϕ; also, that $\sum_0^{\phi} P_1 \cos \phi_1$ is the total external shearing force or loading on the same joint. A similar observation applies to M' (Eq. 6*a*). By referring to Eqs. 4, 5, and 6, in connection with the procedures by which Eqs. 14, 15, and 16 were established, it will be clear that the summations involving M' or P_1 such as $\sum_0^{l} \frac{\cos \phi}{A} \left(\sum_0^{\phi} P_1 \sin \phi_1 \right) \Delta L$ are to be extended from the point of application only of each P_1 to l. There must be in effect, therefore, if not in general form, a summation for each load, P_1, P_2, etc., beginning at its point of application and extending to the further end of the span.

Vertical Loading.

Most reinforced concrete arches are not only symmetrical, but their loading is usually vertical, making $\alpha_1 = \alpha_2 = \text{etc.} = 90^\circ$ and $90^\circ + \phi_1 = \phi$ or $\phi_1 = \phi - 90^\circ$. Hence for that condition:

$$\sin \phi_1 = -\cos \phi \text{ and } \cos \phi_1 = \sin \phi \quad . \quad . \quad . \quad 44c.$$

These values may conveniently be used in Eqs. 40 and 44, but no other changes are required for vertical loading.

In consequence of Eq. 44*c*, if in Eqs. 40 and 44 there is but a single force P_1:

$$\sum_0^{l} \frac{\sin \phi}{A} \left(\sum_0^{\phi} P_1 \sin \phi_1 \right) \Delta L = -P_1 \sum_{a_1}^{l} \frac{\sin \phi \cos \phi}{A} \Delta L . \quad 44d.$$

$$\sum_0^{l} \frac{\cos \phi}{A'} \left(\sum_0^{\phi} P_1 \cos \phi_1 \right) \Delta L = P_1 \sum_{a_1}^{l} \frac{\sin \phi \cos \phi}{A'} \Delta L . \quad 44e.$$

$$\sum_0^{l} \frac{\cos \phi}{A} \left(\sum_0^{\phi} P_1 \sin \phi_1 \right) \Delta L = -P_1 \sum_{a_1}^{l} \frac{\cos^2 \phi}{A} \Delta L \quad . \quad 44f.$$

$$\sum_0^{l} \frac{\sin \phi}{A'} \left(\sum_0^{\phi} P_1 \cos \phi_1 \right) \Delta L = P_1 \sum_{a_1}^{l} \frac{\sin^2 \phi}{A'} \Delta L . \quad . \quad 44g.$$

Furthermore, since $\cos (90^\circ - \phi) = -\cos (90^\circ + \phi)$, the

summation $\sum \frac{\sin\phi\cos\phi}{A}\Delta L$ between the symmetrical limits a_1 and $l - a_1$ will be zero. Again, because $\cos(180 - \phi) = -\cos\phi$ there will result for any load for which a_1, a_2, etc., is less than $\frac{1}{2}l$:

$$-P_1\sum_{a_1}^{l}\frac{\sin\phi\cos\phi}{A}\Delta L = P_1\sum_{0}^{a_1-1}\frac{\sin\phi\cos\phi}{A}\Delta L \quad . \quad 44h.$$

For the same reason:

$$P_1\sum_{a_1}^{l}\frac{\sin\phi\cos\phi}{A'}\Delta L = -P_1\sum_{0}^{a_1-1}\frac{\sin\phi\cos\phi}{A'}\Delta L \quad . \quad 44k.$$

Again, when a_1, a_2, etc., is greater than $\frac{1}{2}l$, the summation of Eq. 44*d* is also expressed by Eq. 44*h*, since the symmetrical terms of the summation (symmetrical in reference to the crown of the arch) cancel each other in pairs, as already observed.

For the same reason, when a_1, a_2, etc., is greater than $\frac{1}{2}l$, the summation of Eq. 44*e* is expressed by Eq. 44*k*.

In the use of Eqs. 44*h* and 44*k*, it must be carefully remembered that ϕ is an acute angle. With the vertical loads supposed, Eq. 6*a* shows:

$$M' = \sum_{0}^{\phi} P_1(x - a_1) \quad . \quad . \quad . \quad . \quad . \quad 44l.$$

If a single load, P_1, rests on the arch, the term $\sum_{0}^{l}\frac{M'y\Delta L}{I}$ in Eq. 40 may be written:

$$\sum_{0}^{l}\frac{y\Delta LM'}{I} = P_1\sum_{a_1}^{l}\frac{(x-a_1)\,y\Delta L}{I} \quad . \quad . \quad 44m.$$

The second member of Eq. 44*m* is to be used in computations to be made hereafter.

Under the same supposition of a single vertical load the numerator of Eq. 41 becomes:

$$\sum_{0}^{l}\frac{M'\Delta L}{I} = P_1\left(\sum_{a_1}^{l}\frac{x\Delta L}{I} - a_1\sum_{a_1}^{l}\frac{\Delta L}{I}\right) \quad . \quad 45.$$

These equations will greatly simplify and shorten the treatment of symmetrical arch ribs with vertical loading, as will be shown further on.

Art. 97.—Moment of Inertia—Intensity of Stress.

Moment of Inertia I

If the rib is of two materials, like reinforced concrete, the moment of inertia I must be written accordingly.

In order to write the value of I for a reinforced concrete rib, let A_2 represent the area of the steel section divided into the two parts $r A_2$ in the lower half of the rib and $(1 - r) A_2$ in the upper half. Assuming the cross-section of the rib to be rectangular, let the dimensions of that cross-section be as shown in Fig. 3, the line $N S$ being the neutral axis, or line of no stress, of the section. The notation of Fig. 3 is self-explanatory, but it may be well to state that d_2 is the distance from the neutral axis to the centre of gravity of $(1 - r) A_2$, while $h_2 - d_2$ is the distance from the neutral axis to the centre of gravity of $r A_2$. If, then, the coefficient or modulus of elasticity of steel be represented by E_2, while E_1 represents that of concrete, and if

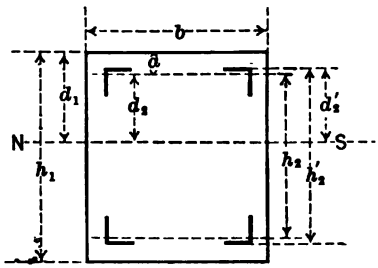


Fig. 3.

$$\frac{E_2}{E_1} = e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 46.$$

the distance, d_1, from the upper surface of the rib to the neutral axis will take the value:

$$d_1 = \frac{\frac{1}{2} h_1^2 + e \frac{A_2}{b} (a + r\, h_2)}{h_1 + e \frac{A_2}{b}} \quad . \quad . \quad . \quad . \quad 47.*$$

In this equation the tensile resistance of the concrete is given full value.

*See Appendix II.

If the tensile resistance of the concrete be neglected, the value of d_1 becomes:

$$d_1 = -e\frac{A_2}{b} \pm \sqrt{\left(e\frac{A_2}{b}\right)^2 + 2e\frac{A_2}{b}\left(a + r\,h_2\right)} \quad . \quad 48.*$$

If the steel should be found wholly on one side or the other (*i.e.*, top or bottom) of the rib, $r = 0$, or 1, as the case may be.

Similar formulæ may be written for other forms of cross-section than the rectangle.

It is obvious that if the cross-section of both steel and concrete is symmetrically disposed about a horizontal line at mid-depth of the rib, the neutral axis will coincide with that mid-depth line, making

$$d_1 = \tfrac{1}{2}\,h_1.$$

This is shown by Eq. 47, because r is then $\frac{1}{2}$ and $a + \frac{1}{2}\,h_2 = \frac{1}{2}\,h_1$.

After d_1 has been determined, the moment of inertia I of the combined cross-section about the neutral axis, now known, must be found.

If the radius of curvature due to bending only is R, then it must be remembered that the moment of the internal stresses of any normal section is:

$$M = \frac{E\,I}{R},$$

for each part of the section. Hence, the quantity $E\,I$ for the reinforced concrete section may be written:

$$E_1\,I_1 + E_2\,I_2 = E_1\,(I + e\,I_2) \quad . \quad . \quad . \quad 49.$$

It is convenient to compute I as if it covered the entire normal section, *i.e.*, not only the concrete part but the area of the steel section; thus giving to Eq. 49 the following form:

$$E_1\,I_1 + E_2\,I_2 = E\,[I + (e - 1)\,I_2] \quad . \quad . \quad . \quad 50.$$

Eqs. 49 and 50 are perfectly general, and they are applicable to ribs of any section.

In Eq. 50, I is the moment of inertia of the entire cross-section of the rib about the neutral axis.

*"Elasticity and Resistance of Materials," by Wm. H. Burr, pp. 639, 640.

If the width or breadth of a rectangular rib is b, and h_1 is its depth, then if $d_1 = \frac{1}{2} h_1$:

$$I = \frac{b h_1^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 51.$$

Assuming, as is usually the case, that the steel is symmetrically disposed, there may be written:

$$I_2 = 2 \left(\frac{A_2}{2} d_1^2 \right) = A_2 d_1^2 \quad . \quad . \quad . \quad . \quad . \quad 52.$$

Inasmuch as the area of steel seldom exceeds 2 per cent. of the entire section, and is often as small as $\frac{1}{2}$ per cent., each part of it may be considered concentrated as its centre of gravity. This greatly simplifies finding the value of I_2.

The moment of inertia of the combined rectangular section then becomes:

$$I + \left(e - 1\right) I_2 = \frac{b h_1^3}{12} + \left(e - 1\right) A_2 d_1^2 \quad . \quad . \quad 53.$$

If these equations are applied to one unit (foot) width of rib, $b = 1$, and A_2 is the area of normal section of steel belonging to a unit width of rib.

These formulæ enable all computations of stresses in any reinforced concrete rib to be made when the bending moment M is known.

In the case of a plain concrete rib, the moment of inertia I is simply as given in Eq. 51. If the rib is of structural steel, the moment of inertia I is to be found precisely as for any steel cross-section, for which the neutral axis passes through its centre of gravity.

Intensities of Stress.

Expressions for the intensities of stress in either the concrete or the steel may readily be written. For this purpose,

Let k_1 = (greatest) intensity of stress in the concrete at distance d_1 from the neutral axis.

k^1 = (greatest) intensity of stress in the concrete at distance $(h_1 - d_1)$ from the neutral axis.

c = (greatest) intensity of stress in the steel at distance d_2 from the neutral axis.

The sign of the bending moment M, Eq. 6, Art. 94, will show whether k^1, k_1, or c is tension or compression. A negative moment will make k_1 and c compression, and hence, k^1 tension.

The intensity k_1 will be:

$$k_1 = \frac{M}{\frac{b\,d_1^3}{3} + \frac{b\,(h_1 - d_1)^3}{3\,d_1} + e\,\frac{d_1^3\,A_2}{d_1}\left[(1 - r) + r\left(\frac{h_2}{d_2} - 1\right)^2\right]} \quad . \quad 54.^*$$

$$k' = \frac{h_1 - d_1}{d_1}\,k_1; \text{ and } c = e\,\frac{d_2}{d_1}\,k_1 \quad . \quad . \quad . \quad 55.$$

Furthermore, the intensity of stress in the steel of the kind opposite to c is:

$$t_1 = \frac{h_2 - d_2}{d_2}\,c = e\,\frac{h_2 - d_2}{d_1}\,k_1 \quad . \quad . \quad . \quad 56.$$

The preceding equations recognize the tensile resistance of the concrete, but if that tensile resistance be neglected, the intensity k_1, now compression, becomes:

$$k_1 = \frac{M}{\frac{b\,d_1^3}{3} + e\,\frac{d_1^3}{d_1}\,A_2\left[(1 - r) + r\left(\frac{h_2}{d_2} - 1\right)^2\right]} \quad . \quad . \quad 57.$$

In this case, $k^1 = 0$, but otherwise Eqs. 55 and 56 are unchanged.

Although the preceding equations belong to rectangular sections, it is clear that similar equations for any form of section may easily be written.

If the steel reinforcement is wholly on the tension side, $r = 1$ in Eq. 57, and:

$$k_1 = \frac{M}{\frac{b\,d_1^3}{3} + e\,\frac{A_2}{d_1}\,(h_2 - d_2)^2} \quad . \quad . \quad . \quad . \quad 58.$$

Bending Moment M.

In all the preceding equations, the moment M is given by Eq. 6, Art. 94, in which M_0, H, and V are found by Eqs. 29, 30, and 32 of Art. 95.

*Op. cit., pp. 642, 643.

When the loading is vertical, $\alpha_1 = \alpha_2 = \text{etc.} = 90°$, then $\cos \alpha_1 = \cos \alpha_2 = \text{etc.} = 0$ and $\sin \alpha_1 = \sin \alpha_2 = 1$. Hence:

$$M = M_0 + H y - V x + \sum_0^{\phi} P_1 (x - a_1) \quad . \quad 58a.$$

Normal Compressive Stress at Any Section.

The total normal stress N on any normal section of an arch rib, given by Eq. 4 of Art. 94, is uniformly distributed over the sectional area of the rib, except that in case of reinforced concrete the uniform intensity of the stress on the steel section A_2 is e $\left(= \frac{E_2}{E}\right)$ times as great as on the concrete section. Hence if p_1 is the uniform intensity of compression due to N, the direct normal stress, then A, being the total normal sectional area of the rib:

$$p_1 A + (e - 1) p_1 A_2 = N$$

Or:

$$p_1 = \frac{N}{A + (e - 1) A_2} \quad . \quad . \quad . \quad . \quad . \quad 59.$$

The greatest intensity of compression in the concrete section will then be, by Eqs. 54, 57, or 58 and 59:

$$p = k_1 + p_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad 60.$$

While the least intensity will be:

$$p^1 = k_1 - p_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad 61.$$

The greatest compression in the steel will be, by the aid of Eq. 55:

$$c = e p_1 + e \frac{d_2}{d_1} k_1 = \frac{e N}{A + (e - 1) A_2} + e \frac{d_2}{d_1} k_1 \quad . \quad 62.$$

The greatest tension in the steel in the same section will be, remembering Eq. 56:

$$t = e p_1 - e \frac{h_2 - d_2}{d_1} k_1 = \frac{e N}{A + (e - 1) A_2} - e \frac{h_2 - d_2}{d_1} k_1 \quad . \quad 63.$$

These equations complete the analysis for the direct stresses of compression and tension in any section of a reinforced concrete arch.

For plain concrete or any other single material it is only necessary to make $e = 1$ or $e - 1 = 0$ in the Eqs. 59, 62, and 63.

Vertical Loading.

If the rib carries vertical loading, $\alpha_1 = \alpha_2 = \text{etc.} = 90°$ and $\sin 90° = 1$, $\cos 90° = 0$. Also, as stated in the preceding article, $\sin \phi_1 = -\cos \phi$, and $\cos \phi_1 = \sin \phi$. Hence, by Eq. 4 of Art. 94:

$$N = H \sin \phi + \left(V - \sum_0^{\phi} P_1\right) \cos \phi \quad . \quad . \quad 64.$$

This value of N is to be used in the preceding equations for p_1, c, and t when the loading is vertical.

The bending moment M for vertical loading is given by Eq. 58*a*.

Shearing Stress in Any Normal Section.

The general value of the total shearing stress S in any normal section of the arch rib is given by Eq. 5, Art. 94. This shearing stress is assumed to be uniformly distributed over the normal section A, which includes the steel section A_2 in the case of reinforced concrete.

When the rib is of solid reinforced concrete it may seriously be doubted whether the steel should be considered as relieving the concrete of any material amount of shear unless the imbedded steel is designed as a braced rib in itself. If the steel is not so designed the intensity s of the shear in the concrete will be:

$$s = \frac{H \cos \phi - V \sin \phi + \sum_0^{\phi} P_1 \cos \phi_1}{A} \quad . \quad . \quad 65.$$

The total sectional area A, which includes A_2, is written, as it is reasonable to suppose that the steel section is at least equivalent to an equal concrete section in resisting shear.

Should the steel reinforcement be designed as a trussed member or braced rib, the amount of shear which it will carry will depend upon that design to such an extent that the amount cannot be determined until the design is known; it may be designed to carry the total shear. The shear which the steel does not carry must evidently be carried by the concrete. It may be observed that it is difficult to arrange the combined concrete and steel so as to make each part carry satisfactorily its computed proper portion. In most cases it is probably best to design the work so that either the concrete or the steel shall afford sufficient capacity to carry all the shear.

When all the loading is vertical, $\cos \phi_1 = \sin \phi$ in Eq. 65, or:

$$s = \frac{H \cos \phi - \sin \phi \left(V - \sum_0^{\phi} P_1\right)}{A} \quad . \quad . \quad . \quad 66.$$

In some cases it may be advisable to determine the shear on some other than the normal section of the rib, as on a vertical plane normal to the vertical axial plane. It is only necessary to determine the resultant stress along such a plane in the same general manner as that employed for the normal section; then find the area of the section in question and divide the former by the latter, remembering the general observations regarding the steel reinforcement made above.

Art. 98.—Application of Formulæ to a Reinforced Concrete Rib.

The rib used in the application of these formulæ is shown on Plate IV. It is taken from a bridge already constructed. The span of the rib from centre to centre of springing joints is 151.2 ft., and the rise of the centre line of the rib at the crown is 47 ft. The curve of the centre-line of the rib is circular, but two centred, the radius of the central part being 78.2 ft., and that of each end part 106.5 ft. The uniform width of the rib is 3.25 ft. The depth of rib at the crown is 3.5 ft., and 7.5 ft. at the springing joint.

There are 13 loads applied to the rib by means of spandril

columns, placed 12.25 ft. from centre to centre. The dead loads of the structure and the moving loads are as follows:

Dead Load.	*Live Load.*	*Dead Load.*
P_1 = 75,600 lbs.	24,000 lbs.	P_8 = 36,500 lbs.
P_2 = 53,300 "	24,500 "	P_9 = 44,000 "
P_3 = 49,800 "	24,500 "	P_{10} = 47,400 "
P_4 = 47,400 "	24,500 "	P_{11} = 49,800 "
P_5 = 44,000 "	24,500 "	P_{12} = 53,300 "
P_6 = 36,500 "	24,500 "	P_{13} = 75,600 "
P_7 = 36,500 "		

The points of application of the loads fix the lengths (ΔL) of the sections into which the rib is divided. The first load, P_1, acts on the rib at the horizontal distance a_1 = 2.1 ft. from the joint, where the centre-line of the rib cuts the springing joint, which point is taken as the origin of the co-ordinates x and y, and a and b. The lengths ΔL along the centre-line of the rib are the distances between the points of application of the loads P_1, P_2, P_3, etc., and the distances of P_1 and P_{13} from the ends of the centre-line.

The co-ordinates x and y locate the normal sections of the rib at the mid-points of the length ΔL, *i.e.*, midway between the points of application of each two adjacent loads. This part of the procedure is based upon the tacit assumption that the state of stress is uniform throughout each length ΔL, and it involves a corresponding approximation.

All dimensions, even for cross-sectional areas of steel and for moments of inertia, are taken in feet. Hence areas of sections will be given in square feet, and the modulus of elasticity for steel will be 144 × 30,000,000 = 4,320,000,000. The modulus for concrete must be similarly taken, *i.e.*, 144 × 2,000,000 = 288,000,000. The ratio e will thus become:

$$e = \frac{30}{2} = 15.$$

The ratio n between the moduli of elasticity for normal and shearing stresses will be taken as .4.

The subscript 0 applies to the springing joint, but A'_0 or I'_0 indicates the area of normal rib section or moment of inertia midway of the ΔL_0 at the end of the rib, and similarly for other letters affected with the prime and zero. The subscripts 1, 2, 3, etc., indicate the joints, loads, etc., numbered consecutively from the end of the rib.

Fig. 4 shows the cross-section of the rib with the steel reinforcement. The depth of the rib is represented by h_0, h'_0, h_1, h_2, etc., as given below. Fig. 4 shows the cross-section where the depth is $h_4 = 4.4$ feet.

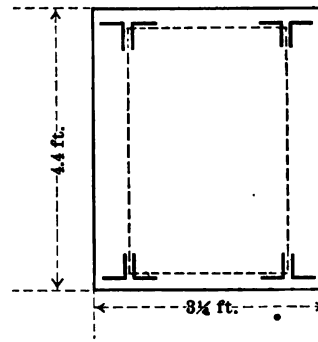


FIG. 4.

The values of the angles ϕ_0, ϕ_0', ϕ', ϕ'', etc., of inclination of the normal section of the rib to a horizontal line at the mid-points of ΔL are:

$$\begin{aligned} \phi_0 &= 30^\circ\ 25' & \phi''' &= 56^\circ\ 45' \\ \phi_0' &= 32\ \ 00 & \phi^{iv} &= 66\ \ 45 \\ \phi' &= 38\ \ 35 & \phi^{v} &= 76\ \ 15 \\ \phi'' &= 48\ \ 15 & \phi^{vi} &= 85\ \ 5 \end{aligned}$$

The inclinations of the normal joints in the second half of the arch are not needed, as they would be supplements of the angles of inclination in the first half.

The lengths ΔL, the total areas of normal sections of the rib, and the areas of steel section (about ½ per cent. of the total sectional area) are as follows:

Values of ΔL.	*Total Section.*	*Steel Section.*
$\Delta L_{13} = \Delta L'_0 = 4.0$ ft.	$A_0 = 24.4$ sq. ft.	$A_{20} = .125$ sq. ft.
$\Delta L_{12} = \Delta L_1 = 19.9$ "	$A'_0 = 24.2$ " "	$A'_{20} = .125$ " "
$\Delta L_{11} = \Delta L_2 = 16.5$ "	$A_1 = 22.8$ " "	$A_{21} = .1$ " "
$\Delta L_{10} = \Delta L_3 = 14.6$ "	$A_2 = 19.2$ " "	$A_{22} = .1$ " "
$\Delta L_9 = \Delta L_4 = 13.3$ "	$A_3 = 16.6$ " "	$A_{23} = .08$ " "
$\Delta L_8 = \Delta L_5 = 12.7$ "	$A_4 = 14.3$ " "	$A_{24} = .08$ " "
$\Delta L_7 = \Delta L_6 = 12.3$ "	$A_5 = 11.7$ " "	$A_{25} = .06$ " "
93.3 "	$A_6 = 11.4$ " "	$A_{26} = .06$ " "

The moments of inertia, depths of rib, and the co-ordinates x, y, a, and b are shown below:

Moment of inertia.	*Depth of rib.*
$I_0 = 135.8$	$h_0 = h_{14} = 7.5$ ft.
$I_0' = 132.2$	$h_0' = h_{13} = 7.46$ "
$I_1 = 106.8$	$h_1 = h_{12} = 7.0$ "
$I_2 = 65.86$	$h_2 = h_{11} = 5.9$ "
$I_3 = 41.9$	$h_3 = h_{10} = 5.1$ "
$I_4 = 27.33$	$h_4 = h_9 = 4.4$ "
$I_5 = 14.65$	$h_5 = h_8 = 3.6$ "
$I_6 = 13.53$	$h_6 = h_7 = 3.5$ "

$x_0^1 = 1.05$ ft.	$y_0' = 1.6$ ft.	$a_1 = 2.1$ ft.	$b_1 = 3.2$ ft.
$x_1 = 8.$ "	$y_1 = 11.20$ "	$a_2 = 14.35$ "	$b_2 = 18.4$ "
$x_2 = 20.4$ "	$y_2 = 24.4$ "	$a_3 = 26.6$ "	$b_3 = 29.8$ "
$x_3 = 32.4$ "	$y_3 = 34.$ "	$a_4 = 38.85$ "	$b_4 = 37.8$ "
$x_4 = 44.8$ "	$y_4 = 40.8$ "	$a_5 = 51.1$ "	$b_5 = 43.$ "
$x_5 = 57.2$ "	$y_5 = 45.$ "	$a_6 = 63.35$ "	$b_6 = 46.$ "
$x_6 = 69.6$ "	$y_6 = 46.8$ "	$a_7 = 75.6$ "	$b_7 = 47.$ "
$x_7 = 81.6$ "	$y_7 = 46.8$ "	$a_8 = 87.85$ "	$b_8 = 46.$ "
$x_8 = 94.$ "	$y_8 = 45.$ "	$a_9 = 100.1$ "	$b_9 = 43.$ "
$x_9 = 106.4$ "	$y_9 = 40.8$ "	$a_{10} = 112.35$ "	$b_{10} = 37.8$ "
$x_{10} = 118.8$ "	$y_{10} = 34.$ "	$a_{11} = 124.6$ "	$b_{11} = 29.8$ "
$x_{11} = 130.8$ "	$y_{11} = 24.4$ "	$a_{12} = 136.85$ "	$b_{12} = 18.4$ "
$x_{12} = 143.2$ "	$y_{12} = 11.2$ "	$a_{13} = 149.1$ "	$b_{13} = 3.2$ "
$x_{13} = 150.15$ "	$y_{13} = 1.6$ "		
$x_{14} = 151.2$ (span)			

The results of the successive applications of a single force P to each point of loading throughout the entire rib will next be found.

The quantity M^1 given by Eq. 6a will now take the form:

$$M^1 = P(x - a_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad 67.$$

Consequently the terms in Eqs. 40, 41, and 44, in which M^1 appears, will become:

$$\sum_0^l \frac{M' y \Delta L}{I} = P\left[\sum_{a_1}^l \frac{x y \Delta L}{I} - a_1 \sum_{a_1}^l \frac{y \Delta L}{I}\right] \quad 68.$$

$$\sum_0^l \frac{M' \Delta L}{I} = P\left[\sum_{a_1}^l \frac{x \Delta L}{I} - a_1 \sum_{a_1}^l \frac{\Delta L}{I}\right] \quad . \quad . \quad 69.$$

$$\sum_0^l \frac{M' x \Delta L}{I} = P\left[\sum_{a_1}^l \frac{x^2 \Delta L}{I} - a_1 \sum_{a_1}^l \frac{x \Delta L}{I}\right] . \quad 70.$$

The following tabulations illustrate the application of the preceding data to Eqs. 38, 39, 40, 41, 42, 43, and 44 for the determination of the quantities a, b, c, a', b', c', a'', b'', and c''.

y	$\frac{y \Delta L}{I}$	$\frac{y^2 \Delta L}{I}$	$\frac{x y \Delta L}{I}$	x
1.6	.0484	.0775	.0508	1.05
11.3	2.1055	23.7925	16.8442	8.0
24.7	6.1881	152.8458	126.2370	20.4
34.1	11.8822	405.1813	384.9817	32.4
40.8	19.8406	809.4977	888.8602	44.8
44.8	38.8371	1739.903	2221.4833	57.2
46.8	42.5454	1991.1252	2961.1605	69.6
....	121.4474	5122.4230	3471.7055	81.6
....	.....	.....	3650.6968	94.0
....	.....	.....	2111.0403	106.4
....	.....	.....	1411.5994	118.8
....	.....	.....	809.4022	130.8
....	.....	.....	301.5119	143.2
....			7.2702	150.15
			18362.8442	

a	$\Sigma_a^l \frac{x y \Delta L}{I}$	$-a\Sigma_a^l \frac{y \Delta L}{I}$	$\Sigma_a^l \frac{x \Delta L}{I}$	$\Sigma_a^l \frac{M' \Delta L}{I}$	$\Sigma_a^l \frac{M' y \Delta L}{I}$
.....	18362.844	.....	.....	.....	.....
2.1	18362.793	— 509.977	464.967	452.104	17852.816
14.35	18345.949	— 3454.633	463.476	378.25	14891.316
26.60	18219.712	— 6239.111	458.365	307.049	11980.601
38.85	17834.73	— 8650.768	447.075	239.611	9183.962
51.1	16945.87	— 10364.664	425.23	177.199	6581.206
63.35	14724.387	— 10389.02	375.703	123.13	4335.367
75.6	11763.226	— 9181.469	312.43	79.744	2581.757
87.85	8291.52	— 6931.629	238.248	47.122	1359.891
100.1	4640.823	— 4010.607	156.76	26.444	630.216
112.35	2529.783	— 2272.279	105.019	13.390	257.245
124.6	1118.184	— 1039.538	63.623	5.42	78.646
136.85	308.782	— 294.912	30.853	1.213	13.87
149.10	7.27	— 7.216	4.545	.275	.054

a	$\Sigma_a^l \frac{\Delta L}{I}$	$-a\Sigma_a^l \frac{\Delta L}{I}$	$\Sigma_a^l \frac{x^2 \Delta L}{I}$	$-a\Sigma_a^l \frac{x \Delta L}{I}$	$\Sigma_a^l \frac{M' x \Delta L}{I}$
.....	6.1557	.....	41417.788	.....	.....
2.1	6.12544	− 12.8634	41417.755	− 976.431	40441.324
14.35	5.93911	− 85.2262	41405.830	− 6650.881	34754.949
26.60	5.68858	− 151.3162	41301.569	− 12192.509	29109.06
38.85	5.34013	− 207.464	40935.780	− 17368.864	23566.916
51.1	4.85384	− 248.0312	39959.776	− 21729.253	18230.523
63.35	3.98694	− 252.5726	37123.417	− 23800.785	13322.632
75.6	3.07785	− 232.6855	32719.637	− 23619.708	9099.929
87.85	2.16876	− 191.1256	26766.410	− 20930.087	5836.323
100.1	1.30186	− 130.3162	19106.482	− 15691.676	3414.806
112.35	.81557	− 91.6293	13601.208	− 11798.885	1802.323
124.6	.46712	− 58.2032	8683.375	− 7927.426	755.949
136.85	.21659	− 29.6403	4397.150	− 4222.233	174.917
149.1	.03026	− 4.5117	682.207	− 677.66	4.547

ϕ	a	$\Sigma_0^{a-1} \frac{\sin\phi\cos\phi\,\Delta L}{A}$	$-\Sigma_0^{a-1} \frac{\sin\phi\cos\phi\,\Delta L}{A'}$	$-\Sigma_a^l \frac{\cos^2\phi\,\Delta L}{A}$	$\Sigma_a^l \frac{\sin^2\phi\,\Delta L}{A^1}$
32°	.....	.....	.....	.....	.....
38° 35′	2.1	.0693	− .0743	− 2.7195	8.8585
48° 15′	14.35	.4702	− .4998	− 2.2170	8.5190
56° 45′	26.60	.8681	− .9267	− 1.8618	8.0407
66° 45′	38.85	1.2459	− 1.3300	− 1.6141	7.4255
76° 15′	51.1	1.5587	− 1.6673	− 1.4798	6.5474
85° 5′	63.35	1.7925	− 1.9179	− 1.4225	5.5232
94° 55′	75.6	1.8783	− 2.0101	− 1.4152	4.4524
103° 45′	87.85	1.7925	− 1.9179	− 1.4078	3.3816
113° 15′	100.1	1.5587	− 1.6673	− 1.3506	2.3575
123° 15′	112.35	1.2459	− 1.3300	− 1.2162	1.4793
131° 45′	124.6	.8681	− .9267	− .9685	.8642
141° 25′	136.85	.4702	− .4998	− .6133	.3859
148° 00′	149.1	.0693	− .0743	− .1109	.0464

The following values, which do not involve the load P, at once, result:

a = − 1.78939 (From Eq. 38).

b = .02367 (From Eq. 39).

$a' = -1.91597$ (From Eq. 41).

$b' = .02534$ (From Eq. 41).

$a'' = -2.2571$ (From Eq. 42).

$b'' = b' = .02534$ (From Eq. 43).

The quantities c, c', and c'' (Eqs. 40, 41, and 44) will each have a separate value for each point of application of P. If P be taken as unity, these values will be:

a	c	c'	c''
2.1 ft.	1.73968	1.86131	2.20395
14.35 "	1.451005	1.55726	1.8942
26.6 "	1.16728	1.26412	1.58664
38.85 "	.89545	.98648	1.28471
51.1 "	.64098	.72953	.99395
63.35 "	.42208	.50693	.7265
75.6 "	.25118	.328306	.49637
87.85 "	.13213	.194001	.31846
100.1 "	.061077	.10887	.18642
112.35 "	.0248	.055127	.098459
124.6 "	.007477	.022114	.041361
136.85 "	.001251	.004994	.009622
149.1 "	.0000097	—.0011322	.000261

The negative and exceedingly small value of c for $a = 149.1$ is evidently a small error due to the approximate character of the method for all exceedingly small values due to placing the load close to the end of the span.

Introducing the above numerical values in Eqs. 44a, 44b, and 32, there will result:

$$M_0 = 222.19\,(c'' - c') - 598.8\,(c' - c) \quad . \quad . \quad . \quad . \quad 71.$$

$$V = 2.93143\,(c'' - c') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 72.$$

$$H = 1.78939\ V - .02367\ M_0 - c \quad . \quad . \quad . \quad . \quad 73.$$

These are the equations of the influence lines for M_0, V, and H, shown on Plate VI.

Eqs. 71, 72, and 73 give, by the aid of the preceding values of c, c' and c'', the following results:

a	M_0	V	H
2.1 ft.	+ 3.299 ft. lbs.	.9944 lb.	− .0396 lb.
14.35 "	+ 11.207 " "	.9876 "	.0501 "
26.6 "	+ 13.676 " "	.9454 "	.1998 "
38.85 "	+ 11.758 " "	.8742 "	.39 "
51.1 "	+ 5.73 " "	.7748 "	.6097 "
63.35 "	− 2.02 " "	.6437 "	.7778 "
75.6 "	− 8.843 " "	.4927 "	.8403 "
87.85 "	− 11.51 " "	.3644 "	.7431 "
100.1 "	− 11.385 " "	.2273 "	.615 "
112.35 "	− 8.534 " "	.127 "	.405 "
124.6 "	− 4.483 " "	.0564 "	.1996 "
136.85 "	− 1.213 " "	.136 "	.0518 "
149.1 "	− 0.5(?) " "	.0004 (?) ft.	.0125 "

These values are for one pound (or other unit load) placed at the points on the rib located by the horizontal co-ordinate a.

The values of H should be equal in pairs symmetrical to a vertical line through the crown of the arch, and they are so with substantial accuracy. Again, the values of V added in symmetrical pairs should make the sum of each such pair equal to unity, and that will be found to be essentially the case.

The above numerical values of M_0, V, and H have been used to construct the influence lines on Plate VI. All the points fall essentially on the smooth curves except the value of M_0 for $a = 87.85$. The value $M_0 = -11.51$ ft.-lbs., given in the tabulation above, was scaled from the plate, the computed values being a little less.

The line $B'\,M\,L\,D'$ represents the numerical values of M_0, found in the second column of the table, the ordinates laid off upward from $B'\,D'$ being positive and those below that line negative. The reversed curve $K\,D'$ represents the end reactions or shears V, for the unit load placed at the points represented by a, as shown in the third column of the table. Finally the curved line $B'\,F\,D'$ represents the values of H shown in the fourth

column of the table for unit loads placed at the points indicated by the horizontal co-ordinates a, as for M_0 and V. These three lines, therefore, constitute complete influence lines for the quantities which they represent. The dotted line $K D'$ intersects the curved influence line for V at the point shown, where it also intersects the vertical line $G F$ drawn through the centre of the span, thus constituting a check on the accuracy of the influence line for V.

The end moment M_0 and the vertical and horizontal forces V and H, for any load at any point, can at once be found by multiplying the above unit value of the same quantity (taken from the preceding table or scaled from Plate VI) by that load.

The general values of the normal or shearing force (N or S), or of the bending moment M, at any point given by equations 4, 5, and 6, may now be determined by simply adapting those equations to vertical loading as indicated by Eq. 44c, and in the paragraph preceding that equation. These operations will enable the influence lines for M, N, and S to be drawn. The complete conditions of stress in the entire rib, or its complete design, may then be treated as set forth in Art. 97.

Art. 99.—Intersection Locus for Reaction Lines.

In finding positions of the moving load which cause the greatest bending moment at any point of an arch rib, it is convenient to resort to a graphical method which involves the construction of the locus of the intersection of the reaction lines belonging to all positions which a single or concentrated load may occupy.

In Fig. 5 a single vertical force or load is represented with its line of action through J and K. The reaction lines $J E$ and $J F$ are supposed to be so drawn as to make the actual bending moment in the rib as found in the preceding article, the ends of the rib being fixed and the moments M_0 at B and M'_0 at D being equal to the products of the components of P along $J E$ and $J F$ by the normals drawn from B and D, respectively, to those lines.

The vertical components or reactions at B and D are V and V', and the common horizontal component of each inclined reaction is represented by H. The problem is to find the locus of J as

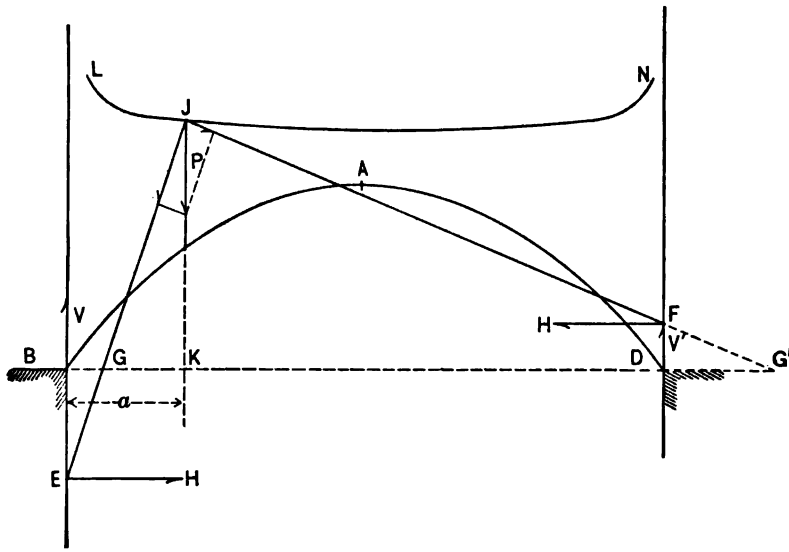


FIG. 5.

the load P is moved from point to point in the span.

The conditions represented in Fig. 5 lead to the following eqautions:

$$M_0 = H \times BE \therefore BE = \frac{M_0}{H}. \quad . \quad . \quad . \quad 74.$$

Taking moments about G:

$$V \times BG = H \times BE \therefore BG = \frac{H}{V} BE = \frac{M_0}{V} \quad . \quad 75.$$

Similarly:

$$M_0' = H \times FD \therefore FD = \frac{M_0'}{H}. \quad . \quad . \quad . \quad 76.$$

Also, taking moments about G^1:

$$V' \times DG' = H \times FD \therefore DG' = \frac{H}{V'} FD = \frac{M_0'}{V'} \quad . \quad 77.$$

By similar triangles:

$$\frac{JK}{GK} = \frac{BE}{BG} = \frac{V}{H} \therefore JK = GK\frac{V}{H}$$

$$\text{Because } GK = a - BG = a - \frac{M_0}{V}.$$

$$JK = \left(a - \frac{M_0}{V}\right)\frac{V}{H} = \frac{Va - M_0}{H}. \quad . \quad . \quad . \quad 78.$$

Eq. 78 is the equation of the intersection locus J for the reaction lines EJ and FJ. That locus is the line LJN. The extremities L and H are indeterminate at the vertical lines through B and D, for when $a = 0$, in Eq. 78, M_0 and H both become zero, making $JK = \frac{0}{0}$.

It is evident from Fig. 5 that the intersection locus may be constructed by finding BG and DG', or BE and FD, for all positions of the load P. In either case the quantities become large for positions of the load near the ends of the span. The horizontal distances BG and DG' will be used in this case.

By introducing the quantities M_0, V, and H established for unit loads in the preceding article in Eqs. 75, 77, and 78, the following numerical results are reached:

a	BG	DG'	JK
2.1 ft.	+ 3.32 ft.	.....	.....
14.35 "	+ 11.35 "	− 89.19 ft.	59.3 ft.
26.6 "	+ 14.47 "	− 79.49 "	57.36 "
38.85 "	+ 13.45 "	− 67.2 "	.....
51.1 "	+ 7.39 "	− 50.09 "	55.51 "
63.35 "	− 3.14 "	− 31.59 "	.....
75.6 "	− 17.94 "	− 17.94 "	55.0 "
87.85 "	− 31.59 "	− 3.14 "	.....
100.1 "	− 50.09 "	+ 7.39 "	55.96 "
112.35 "	− 67.2 "	+ 13.45 "	.....
124.6 "	− 79.49 "	+ 14.47 "	57.55 "
136.85 "	− 89.19 "	+ 11.35 "	59.1 "
149.1 "		+ 3.32 "	.....

In Plate VI, the smooth curve $N\,O\,P$ represents the points established by the vertical distances $J\,K$ above $B\,D$. The points lie in the curve with essential accuracy.

The reaction lines drawn from points located by the distances $B\,G$ and $D\,G'$, from B and D, respectively, to points in the intersection locus determined by the ordinate $J\,K$ are all shown within the limits established by vertical lines through B and D.

Art. 100.—Position and Amount of Loading Required to Give Greatest Bending Moment at Any Joint.

Referring to the intersection locus and the reaction lines constructed on Plate VI, as fully explained in the preceding article, it may at once be observed that if the two curves, $F\,Q$ and $R\,S$, be drawn tangent to the reaction lines as shown on the plate, those curves will be envelopes. These curved envelopes are obviously exact duplicates. It is equally clear that if two lines be drawn tangent to the envelopes from any point in the intersection locus, those two tangents will represent reaction lines for a vertical load, acting through the assumed point on the locus.

The bending moment at any joint in either of the two segments of the rib made by the point of application of a vertical load will be found by simply multiplying the component of the load acting along the reaction line by the normal distance of the centre of the joint in question from the latter line. Therefore the joint located by the intersection of a reaction line with the centre line of the rib will be subject to no bending, as the lever arm will be 0.

If, then, f be the centre of any joint of the rib, let the broken line $c\,f\,d$ be drawn tangent to the right envelope. The reaction line $a\,f\,o$ is already constructed tangent to the left envelope. Any loading placed between c and a will produce a bending moment at the joint f, causing compression in the upper part of the joint and tension in the lower part. On the other hand, any loading to the right of a or to the left of c will produce a bending moment at f, causing tension in the upper part of the

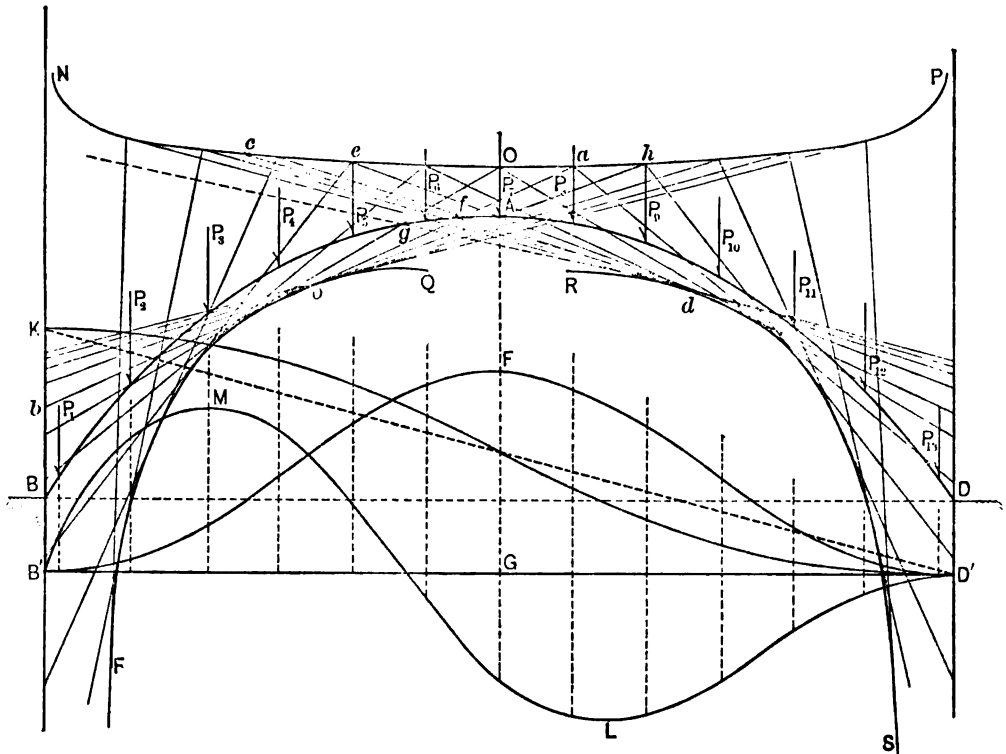


PLATE VI.—Influence and Other Lines for Arch Rib with Fixed Ends.

joint and compression in the lower. It follows, therefore, that if the moments producing compression in the upper part of a joint are called positive, the maximum positive bending moment at *f* will be produced by placing the greatest possible amount of loading between *c* and *a*, and that the greatest negative bending moment at *f* will be produced by removing all load from the segment *c a* and covering the remainder of the span with the greatest possible amount of loading.

Similarly, if it is desired to produce the greatest possible positive bending moment at the crown *A*, *A h*, and *A e*, drawn tangent to the two envelopes, give the segment *e h*, which must be loaded to the greatest extent possible, while the greatest negative moment at the crown will be produced by removing all load from *e h* and covering the remainder of the span.

If the centre of a joint like *g*, at some distance from the crown, be taken, the tangent *g d* to the right envelope does not cut the intersection locus in the left part of the span, but the other tangent *o g* cuts the locus at *o*. This indicates that the greatest positive bending moment at *g* will be produced by loading, to the greatest possible extent, that part of the span extending from *o* to the left end, while the greatest negative moment will be produced by loading the entire right-hand part of the span.

These illustrations show clearly and fully how to determine the amount and position of loading for the greatest bending moment of either kind at any joint by simply drawing tangents from the centre of the joint in question to the two envelopes and observing the segment of the intersection locus made by those tangents produced.

The preceding constructions have been made for vertical loading, but the same general procedure may be employed for any inclined loading, although the resulting constructions and their use would not be so simple as for vertical loads.

Art. 101.—Treatment of the Arch Rib with Hinged Ends, According to the Law of Least Work.

The general treatment of elastic arch ribs, by the aid of the law of least work, as set forth in the preceding articles, is directly applicable to arch ribs with hinged ends with a few changes which simplify analysis.

If the ends of an arch rib are hinged, the end moments, M_0, are each equal to 0. Using the same notation as in Art. 94 the values of the normal stress, the shearing stress, and the bending moment in any normal section of the rib will be given for this case by Eqs. 4, 5, and 6 of Art. 94, after making $M_0 = 0$ in the last of those three equations.

The general expression for the total expended work in the rib due to any load is given by Eq. 7 of the same article precisely as it stands.

Since the end moments, M_0, are each equal to 0, the partial derivative of W, in respect to M_0, will be 0, and Eq. 12 of Art. 95 will consequently disappear. The two partial derivatives of the total expended work W, in respect to the horizontal and vertical components of the reaction, H and V respectively, will be given by Eqs. 11 and 13 of Art. 95, precisely as they stand. Furthermore, the partial derivatives of M, N, and S, in respect to H and V, will also remain precisely as shown between Eqs. 13 and 14 of Art. 95. The two equations of condition, Eq. 14 and Eq. 16 of the same article, will, therefore, be unchanged, while Eq. 15 will disappear for the reason already stated.

The values of a, b, c, a'', b'', c'', will obviously remain exactly the same as shown by Eqs. 17, 18, 19, 21, 22, and 23 of Art. 95, Eqs. 20 disappearing for the same reason that Eq. 15 disappears. The values of b and b'' will not be needed for the reason that M_0 is 0.

The two equations of condition, by which H and V are found, are then the same as Eqs. 24 and 26 of the same article, after making $M_0 = 0$, and they are as follows:

$$H + a\ V + c = 0 \quad \ldots\ldots\ 79.$$

$$H + a''\ V + c'' = 0 \quad \ldots\ldots\ 80.$$

Eliminating H between these two equations, there will result:

$$V = \frac{c'' - c}{a - a''} \quad . \quad . \quad . \quad . \quad . \quad . \quad 81.$$

Then, from Eqs. 79 and 80:

$$H = -aV - c = -a''V - c'' \quad . \quad . \quad . \quad 82.$$

These values of H and V will enable all the stress conditions in the rib to be determined at each and every joint by the aid of Eqs. 4, 5, and 6 of Art. 94.

Eq. 81 is the same as Eq. 30, or Eq. 31, of Art. 95, after making $M_0 = 0$, and the same observation can be made as to H in connection with Eq. 32 of the same article.

In other words, all results required for a rib with hinged ends follow at once from those established for the arch rib with fixed ends by simply making the end or fixing moments equal to zero.

Indeed it would be equally simple to deduce from the analysis of Arts. 94 and 95 the formulæ required for an arch rib with one end fixed and one end hinged, or for a rib with ends fixed but with a hinge at the crown. The establishment of a perfectly

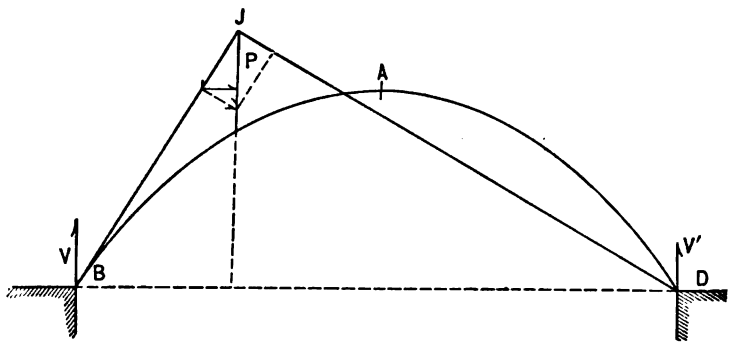


FIG. 6.

general analysis has the marked advantage of including all formulæ for special cases which follow from simple substitution of such quantities as are required for any particular case.

Influence lines for the horizontal stress, H, and for the vertical component, V, of the reaction, may be at once constructed from Eqs. 81 and 82, precisely as shown on Plate VI, for fixed ends, but the procedure has been so fully set forth in the latter case that no further construction need be made for hinged ends. Obviously, since the end bending moments are o, there will be no influence line for such moments.

Similarly, the intersection locus for reaction lines with the corresponding envelopes may be constructed in exactly the same manner as for fixed ends as shown on Plate VI. The general procedure of construction is unchanged; it is simply adapted to the condition of the end moments being o.

Fig. 6 shows the construction of one point, J, on the intersection locus. As the ends of the rib are now hinged the reaction lines, $J\,B$ and $J\,D$, must pass through the centres of the hinges at B and D, the intersection J being in the line of action of the load P. If the distance of the load P, from the left end, B, of the span is a, the following simple analysis results:

$$\frac{J\,K}{B\,K} = \frac{V}{H} \therefore J\,K = B\,K\,\frac{V}{H} = a\,\frac{V}{H} \quad . \quad . \quad . \quad 83.$$

$$V = P\,\frac{l-a}{l} \therefore J\,K = \frac{P}{l}\,\frac{a\,(l-a)}{H} \quad . \quad . \quad . \quad 84.$$

Eqs. 83 and 84 could have been written at once from Eq. 78 and the value preceding it of Art. 99, by making $M_0 = o$. The equation of the intersection locus is Eq. 83, and the curve itself can be constructed precisely as shown on Plate VI.

Finally, lines drawn through the centre of any joint of the rib, tangent to the envelope at one end and extended to the intersection locus at the other, will show the position and amount of loading which must rest upon the rib in order to give either the greatest positive or negative bending moment at that joint, precisely as explained and illustrated for the fixed end rib in Art. 100.

As a matter of practice it will not generally be necessary to construct an envelope nor to find exactly the amount of loading

required to produce the greatest bending moment at any point in the rib, although that procedure should be followed wherever the magnitude of the structure requires the greatest attainable accuracy in computations.

Obviously it may be desirable to compute the bending moments and all stresses due to dead load or own weight and then make a corresponding computation for the moving load effects. Tabulations of each set of moments and stresses will then enable any desired combination to be made with ease in the search for greatest moments and stresses.

CHAPTER IX.

THREE-HINGED ARCH RIB.

Art. 102.—Three-Hinged Arch Rib—Graphical Treatment—Vertical Loading.

WHEN an arch rib has three hinges, as one at the crown and one at each end of the span, it is a statically determinate structure. The bending moment is zero at each hinge, and that condition at the crown supplies the remaining necessary equation for complete statical determination.

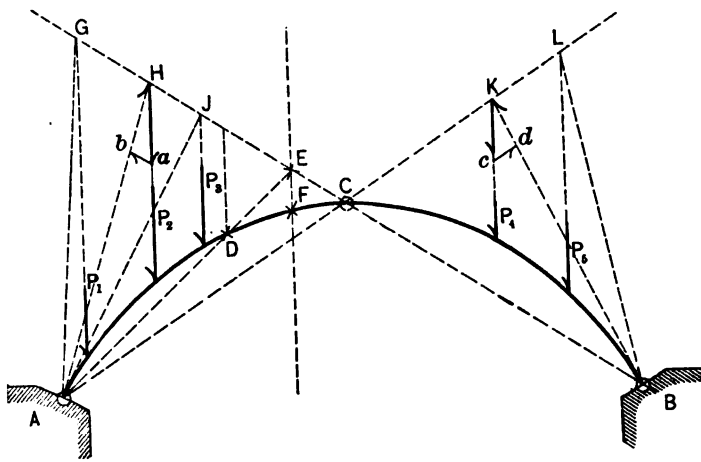


FIG. 1.

Fig. 1 shows a three-hinged rib subjected to vertical loading, the hinge C being at the crown. The loads P_1, P_2, P_3, P_4, and P_5 are supposed each to be of any amount, and to act at any points, but they are vertical.

The line of action of the reaction at B for any load or loads between A and C must necessarily be $B\,C$ extended until it cuts their vertical lines of action as shown in Fig. 1. If those inter-

section points be connected by straight lines with the other extremity of the span A, then will those lines represent the lines of action of the reactions at A, also as shown in Fig. 1.

Obviously the reaction lines for all loads between B and C will be drawn from B to intersection points above C on $A\ C$ extended. Hence the extensions upward from C of $A\ C$ and $B\ C$ will be the reaction intersection lines for all possible loads carried by the rib.

If Ha be laid off to scale from the intersection point H and if ab be parallel to $B\ H$, then will ab be the reaction at B and Hb that at A. The horizontal component of ab is evidently equal to the horizontal component of Hb, but they act in opposite directions. They represent the equal and opposite horizontal reactions at A and B, *i.e.*, the horizontal stress in the closing line of the force polygon. By treating each load in the same manner the total resultant and horizontal reactions become immediately known for all the loading on the structure.

Let it be desired to find the greatest possible bending moment at any point D, Fig. 1, and the amount and position of loading producing it. If the line $A\ D$ be produced to E on the line $B\ C\ G$, it is at once evident that the reaction lines for all vertical loading acting to the left of E will be found upward and to the left of D, while all loads acting between E and C will have reaction lines to the right of and below D. It is equally clear that the last observation applies to all loading on the part $B\ C$ of the rib, as the line $A\ C$ extended will be the reaction line drawn from A for all such loads. Hence if a vertical line be drawn through E locating F on the rib, it is thus shown that all loads between A and F will produce the greatest possible bending moment at D of one sign or kind, while all possible loads placed upon the rib between B and F will produce the greatest moment of the opposite sign or kind at D.

In order, therefore, to produce the greatest possible bending moment at any point in the rib, it is only necessary to draw a reaction line from the nearest springing joint or abutment hinge through that point to the straight line extended from the other abutment through the hinge at the crown and then draw

a vertical line through the intersection point so found. All possible loading on one side of this intersection point will produce the greatest bending moment at the given point on the rib of one sign or kind and all possible loading on the other side of the intersection point will produce the greatest bending moment of the opposite sign or kind.

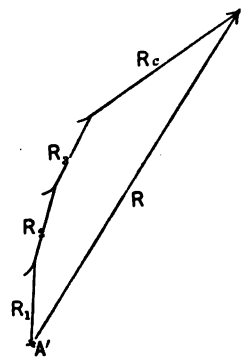


FIG. 2.

If the reactions due to P_1, P_2, P_3, etc., be represented by R_1, R_2, R_3, etc., the resultant reaction at A due to all the loads on the rib may readily be found by a simple force polygon as shown in Fig. 2, the construction of which needs no explanation, except to state that the side R, parallel to $A\,C$, Fig. 1, is the sum of all the reactions at A due to all the loads on $B\,C$ of the same figure. When the resultant R has thus been found the funicular polygon or polygonal frame may readily be constructed in the usual manner so as to pass through the three hinge points A, C, and B of Fig. 1. There will thus be completely determined the true equilibrium polygon or polygonal frame for any assumed system of loads whatever.

The rays of the force polygon from which the polygonal frame is constructed will give the resultant thrust at all points of the rib precisely as has been shown in the graphical treatment of Chapter VII.

The position of the true polygonal frame in reference to the centre line of the rib will enable the bending moments, thrusts, shears, and other stresses to be readily computed at all sections of the rib for any type of the latter which may be under consideration, also as shown in the preceding articles.

If the rib is a curved truss the stresses in all its members may easily be found by the graphical method by a single stress diagram for any given system of loading. If, on the other hand, the rib is a solid structure like a reinforced concrete arch or a steel rib of plate girder section the stress due to bending

and to the direct thrust at any section may be found by the usual method of combining the intensities of stress due to bending and direct thrust.

Inclined Loads.

If the loads instead of being vertical are inclined like those due to normal wind pressure, as shown in Fig. 3, the graphical method is almost as simply applicable as in the case of vertical loads. Inclined loads on both halves of the rib are shown in Fig. 3. The reaction intersection lines are again extensions of *A C* and *B C* above the crown hinge *C*. The line of action of each load is produced precisely as in Fig. 1, to the intersection

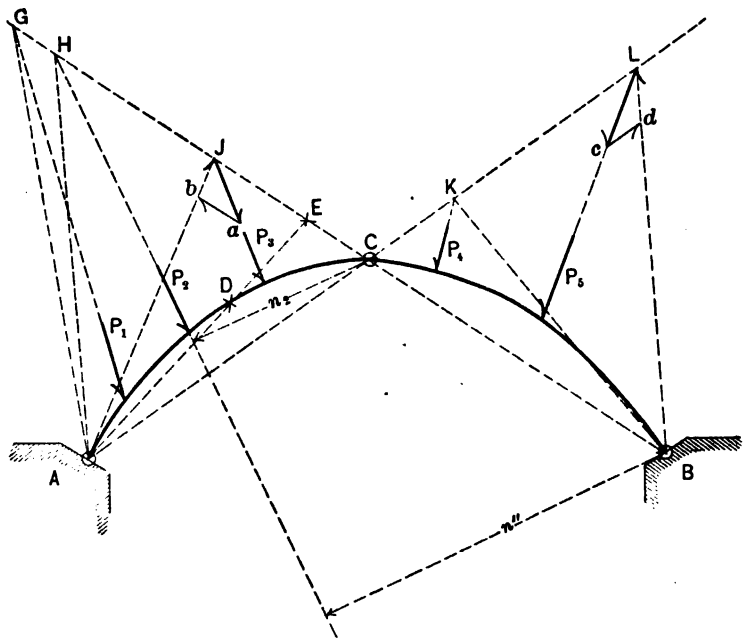


FIG. 3.

lines at *G*, *H*, *J*, *K*, *L*, etc. Lines drawn from those points to *A* and *B* are the lines of action of the reactions for the separate loads indicated.

If *D* is any point of the rib at which it is desired to find

the greatest bending moment, the line $A\,D$ is again prolonged to E. Then, for the same reason as before, all loads whose lines of action are located to the left of E will produce the greatest possible bending moment at D of one sign or kind, while all loads on the rib whose lines of action are located to the right of E will produce bending moments at D of the other sign or kind. Consequently all loads with lines of action to the left of E must act concurrently for the greatest possible bending moment at D of one sign, while all other loads must act concurrently for the greatest possible bending moment at D of the opposite sign. It is thus seen that precisely the same general method is to be followed whether the loads are vertical or inclined. Force triangles like $J\,b\,a$ and $L\,c\,d$ give the two reactions at A and B for each and every single load.

When the resultant reactions at the two ends A and B are found as indicated, the true equilibrium polygon or polygonal frame is to be constructed so as to pass through the three-hinge centres A, C, and B, precisely as set forth for vertical loads. The rays of the force polygon will show the resultant thrusts at all points of the rib also precisely as in the case of the vertical loading. In fact, the procedures to be followed in the determination of thrusts, shears, and bending moments of the actual rib, whether of the solid type or of the truss type, as well as the methods for finding the greatest intensities of stresses due to the thrust and bending moments or shears are identical with those already set forth in the first part of this article.

Art. 103.—Three-Hinged Arch Rib—Analytic Treatment—Vertical Loading.

The analytic treatment of the three-hinged arch rib is exceedingly simple, as the vertical reactions at the two abutment hinges are identical with that for a straight beam of the same span length and carrying the same loads with ends simply supported. Fig. 4 is a skeleton diagram of the rib in question carrying any number of loads at any points. The notation to be used is fully shown in the figure. The crown hinge C is

supposed to be over the centre of the span length $A\,B$, which will be represented by l.

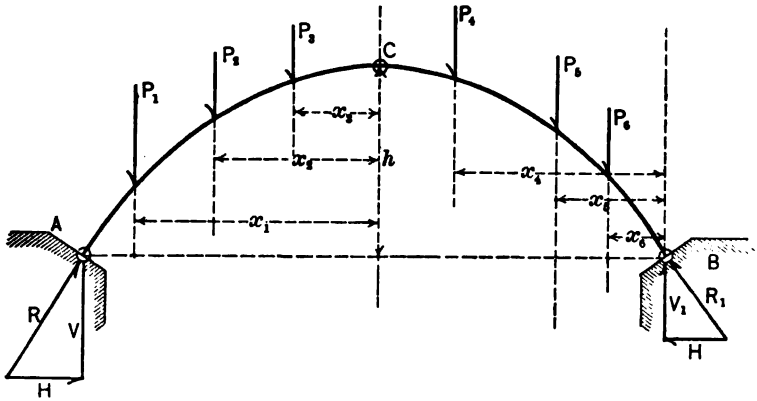


FIG. 4.

The resultant reaction at A is represented by R which has the vertical component V and the horizontal component H. If moments of all the loads and the vertical reaction V be taken about B, the following value of that vertical component will result:

$$V = P_1\frac{x_1}{l} + P_2\frac{x_2}{l} + P_3\frac{x_3}{l} + P_4\frac{x_4}{l} + P_5\frac{x_5}{l} + P_6\frac{x_6}{l} + \frac{P_1+P_2+P_3}{2} \quad . \quad . \quad . \quad . \quad . \quad 1.$$

The horizontal component H does not appear in this equation, as its line of action passes through A and B, making its moment about the latter point zero.

The vertical reaction at B will evidently be:

$$V_1 = P_1 + P_2 + P_3 + P_4 + P_5 + P_6 - V \quad . \quad . \quad 2.$$

The horizontal component H of the resultant reaction R may at once be written by taking moments about the crown hinge C and it will have the following value:

$$H = \frac{l}{2h}V - P_1\frac{x_1}{h} - P_2\frac{x_2}{h} - P_3\frac{x_3}{h} \quad . \quad . \quad . \quad 3.$$

Obviously the resultant reaction R' at B will have the same horizontal component H as the resultant reaction R.

After having found the vertical components V and V_1 of the two reactions at A and B, those reactions will have the following values:

$$\left.\begin{aligned} R &= \sqrt{H^2 + V^2} \\ R_1 &= \sqrt{H^2 + V_1^2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad 4.$$

Having the values of the two reactions R and R_1 or their vertical and horizontal components, moments may then be taken about any or all points of the rib so as to determine bending moments, thrust, shears, and intensities of stress for any type of rib proposed. It will probably be simpler, however, to resort to the graphical method for the construction of force polygons and equilibrium polygons or polygonal frames for the complete treatment of the ribs, as set forth in the preceding article. By following either procedure the complete analysis of an existing arch rib may be made or the design of a new structure completed for any given span and system of loading.

Inclined Loads.

Fig. 3 of Art. 102 will be used in discussing the analytic treatment of inclined loads, the general method of procedure being the same as for vertical loads.

Let it be supposed that normal lines or lever arms are drawn from C and B to the lines of action of all the inclined loads as illustrated by the two lever arms n_2 and n'' for the line of action of the load P_2. The vertical component V of the reaction R at A can then at once be found by taking moments of all the external loading about B. This procedure will give the following value:

$$V = \frac{P_1 n' + P_2 n'' + P_3 n''' + P_4 n^{iv} + P_5 n^{v}}{l} \quad \ldots \quad 5.$$

Evidently a similar expression for the vertical component V_1 of the resultant reaction R_1 at B may be written by supposing the lever arms n', n'', etc., to be drawn from A to the lines of

action of the external loads. It will be necessary to draw such perpendiculars or lever arms, in an actual case, but Eq. 5 may be considered typical for all such operations.

The horizontal component H of the resultant reaction R at A may be found by taking moments of the vertical component V, and the external loads P_1, P_2, and P_3 about the crown hinge C. That operation will give:

$$H = \frac{l}{2h} V - P_1 \frac{n_1}{h} - P_2 \frac{n_2}{h} - P_3 \frac{n_3}{h} \quad . \quad . \quad . \quad 6.$$

The horizontal component H_1 of the reaction R_1 and B, in general different from H, will be found in a similar manner by taking moments about the crown hinge.

Having found the vertical and horizontal components of each reaction, the latter will take the values:

$$\left.\begin{aligned} R &= \sqrt{H^2 + V^2} \\ R_1 &= \sqrt{H_1^2 + V^2_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 7.$$

These expressions make the reactions completely known for every system of external loading. Complete analyses of moments, thrusts, shears, and intensities of stresses in all parts of the structure may then be completed by precisely the same procedures which have already been fully explained both in this and the preceding articles, either by analytic or graphic methods.

It is thus seen that the statically determinate three-hinged arch rib is a structure of exceedingly simple character either to investigate as a completed structure or to design from a prescribed set of data.

CHAPTER X.

Braced Spandrel Arch.

Art. 104.—The Braced Spandrel Arch.

The braced spandrel arch is in reality an arch rib with a horizontal upper chord on which the moving load travels. The web may either be a solid plate or composed of separate web members. The latter type is generally used, as the plate web is not practicable for the lengths of span commonly found.

The end supports are always pins or their equivalents, although the ends might be fixed by anchoring the structure to heavy abutments. It would be perfectly correct, therefore, to consider the ordinary spandrel arch a hinged-end arch rib, and it might be treated graphically or analytically, precisely as has been shown for such ribs, but it is convenient to give it special treatment. The moving load may be either inclined or vertical.

Vertical Moving Loads.

A skeleton diagram of this case is shown in Fig. 1. The pin or hinged ends are found at A and B. The resultant reactions at those points are inclined. With vertical moving loads the horizontal components of the two reactions must always be equal, and they have the same line of action if A and B are at the same elevation. Fig. 1 shows the notation to be employed.

The lines indicated in general by "n," with subscript or otherwise, are lever arms for W_1, W_2, W_3, etc., and web stresses, drawn from the panel points of the two chords and from the intersection points, i, in the upper chord found by prolonging

each lower chord panel until it cuts the upper chord. There will therefore be as many points i as there are lower chord panels,

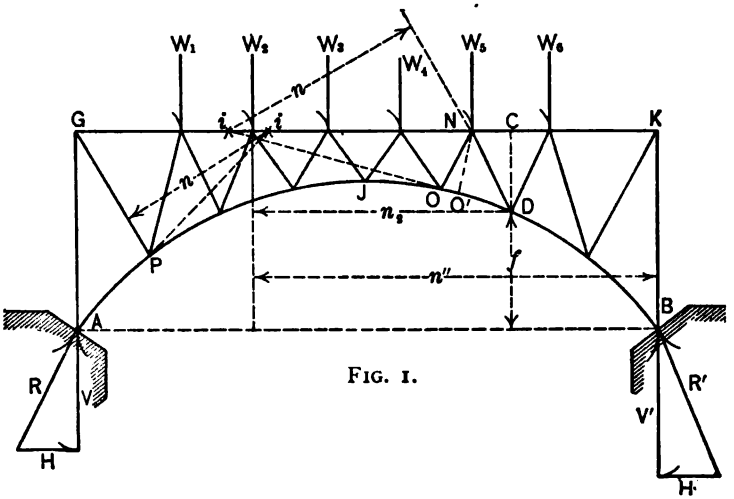


FIG. 1.

and they may be either within or without the limits $G\,K$ of the span.

The following notation will also be employed:

$$C\,D = d;\ A\,B = l;\ G\,C = a.$$

H = horizontal component of reaction R or R^1 at A or B, respectively.

V and V^1 are the vertical components of the same two reactions.

By taking moments about B:

$$V = \frac{W_1 n' + W_2 n'' + \ldots + W_6 n^{vi}}{l} \quad \ldots \quad 1.$$

Hence the simple law of the lever gives the vertical reaction V (or V^1) as in the case of a non-continuous beam simply supported at each end.

Evidently the vertical component V^1 of the reaction at B will be the sum of all the loads from W_1 to W_6 less V, as the sum of V and V^1 must equal the sum of all the loads.

Let T be the stress in any member of the braced arch, as

in the upper chord panel at C, for any arbitrary loading such as has been assumed; then by taking moments about D of all the loads on the left of DC, including the reaction R at A:

$$Va - Hf + Td - W_1 n_1 - W_2 n_2 - W_3 n_3 - \text{etc.} = 0 \quad . \quad 2.$$

Hence:

$$-Hf + Td = -Va + W_1 n_1 + W_2 n_2 + W_3 n_3 + \text{etc.} = M \quad . \quad 3.$$

Eq. 3, the second member of which is completely known, has two unknown quantities, H and T; another equation of condition must therefore be found.

From Eq. 3:

$$T = H\frac{f}{d} + \frac{M}{d} = \frac{Hf + M}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad 4.$$

If F is the area of cross-section of any member in which the stress T exists, the intensity of stress, p, will be:

$$p = \frac{T}{F} = \frac{H}{F}\frac{f}{d} + \frac{M}{Fd} = \frac{Hf + M}{Fd} \quad . \quad . \quad . \quad . \quad 5.$$

If L is the length of a member, the work performed in stressing any member or all members of the structure will be:

$$\frac{p^2 LF}{2E} = \frac{(Hf + M)^2 L}{2EFd^2} \therefore \frac{1}{2E}\sum p^2 LF = \frac{1}{2E}\sum\frac{(Hf+M)^2 L}{Fd^2} \quad 6.$$

Hence, as that work must be a minimum for the proper value of H:

$$\frac{1}{2E}\frac{d(\Sigma p^2 LF)}{dH} = \frac{1}{E}\sum\frac{(Hf^2 + fM)L}{Fd^2} = 0 \quad . \quad . \quad 7.$$

Or,

$$\sum\frac{(Hf^2 + fM)L}{Fd^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad 8.$$

$$\therefore H\sum\frac{f^2 L}{Fd^2} = -\sum\frac{fML}{Fd^2}; \text{ and } H = -\frac{\sum\frac{fML}{Fd^2}}{\sum\frac{f^2 L}{Fd^2}} \quad . \quad 9.$$

Before Eq. 9 can be used, values of the cross-section F of each member of the arch must be assigned, although they depend on the value of H, *i.e.*, upon the resultant stress in each

member. These are tentative or trial values and they may be determined by any convenient approximate method. When these approximate values have been determined and the value of H computed by their aid, they must be revised in accordance with the stresses found by Eqs. 10 and 11. These revised values of F may then be used in Eq. 9 for a revised determination of H. It will not be necessary to repeat this tentative process of computation and revision more than once or twice in any ordinary case, but it must be followed until satisfactory values of F have been found.

In making a practical application of Eq. 9 it is to be remembered that the summation signs are to be used precisely as the equation stands, even if a single load W only rests upon the structure. In that case there would be but a single term Wn in the second member of the value of M, Eq. 3, but that will not change in any way the summation signs in Eq. 9, as those signs relate to, or cover, all members of the structure, *i.e.*, the number of terms represented by each summation sign will be precisely the same as the number of members in the structure.

In the use of Eq. 9, therefore, it will be necessary to determine accurately the value of f for every panel point in the curved chord $A\,J\,B$; every lever arm n for each load W_1, W_2, etc., for every panel point in both the straight upper chord of the arch and the lower curved chord, as well as for each point of support A and B. Furthermore, there must be determined accurately the lever arms n drawn from each intersection point i in the upper chord $G\,K$ to the centre line of each web member prolonged, as $N\,D$ and $G\,P$, as shown in Fig. 1, as well as the normal distances d drawn from each panel point in each chord to the panel in the opposite chord, as $C\,D$ and NO^1. After these normal distances or lever arms have been computed or carefully scaled from an accurate drawing of sufficiently large scale, V may be at once determined by Eq. 1, and that value will make the second term of Eq. 3 a known quantity. The horizontal component H of the resultant reaction R can then at once be computed by Eq. 9, as already indicated. The desired stress T in any member can then be

at once computed from Eq. 3, and it will have one of the following two values which are identical, except that d is used in Eq. 10 and n in Eq. 11:

$$\therefore T = \frac{M + Hf}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 10.$$

$$T = \frac{M + Hf}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 11.$$

Obviously Eq. 10, in which d appears, will give the chord stresses, while Eq. 11, in which n appears, will give the stresses in the web members.

It should be observed that the preceding formulæ have been written in perfectly general forms, and they are applicable whatever may be the curve of the lower chord. They are also applicable precisely as they stand to a polygonal lower chord in which each side of the polygon may be composed of two or more panels. Further, they are also applicable as they stand to a spandrel arch in which the upper chord may be curved as well as the lower, *i.e.*, they are applicable to an ordinary arch rib with hinged ends.

Again, the loads W_1, W_2, W_3 may be considered either as the dead loads or own weight of a structure or as moving loads. In fact, in the general case it will be found convenient to treat the dead loads or own weight separately from the moving loads, and then tabulate the results so as to make any desired combinations whatever.

It is clear that after the horizontal stress H has been determined by the aid of Eq. 9, graphical methods may be employed to determine the stresses in the various members of the spandrel arch instead of the analytic procedures indicated. Indeed the graphical method may be employed in many cases in such way as materially to save labor.

Inclined Moving Loads.

In the preceding section it has been assumed that all moving loads are vertical, whereas the moving loads may act in inclined directions, as shown in Fig. 2.

The case of vertical loading has been treated first for the reason that it is by far the most common, while at the same time the formulæ are applicable to inclined loading without any change. As shown in Fig. 2, if the loading is inclined, the lever arms n_1, n_2, etc., n', n'', etc., are inclined and are each

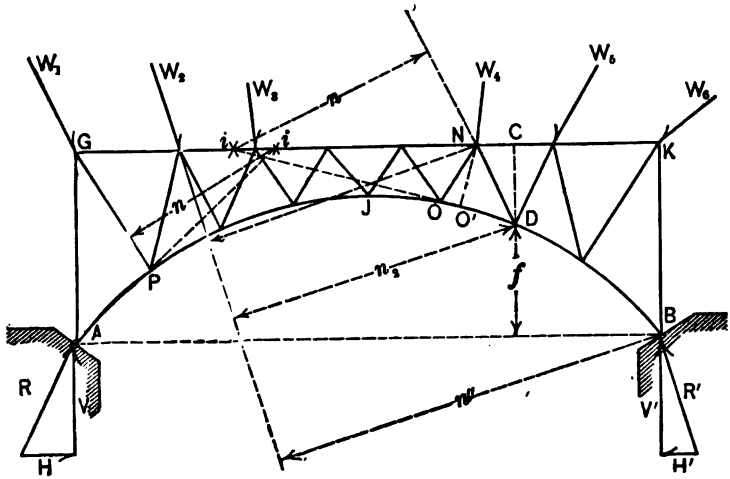


FIG. 2.

measured normal to the inclined line of action of the particular load to which it belongs, and they are each measured from the line of action of each inclined load either to an intermediate point of the lower or upper chord in the first instance, or to the end of the span B in the second instance. The lever arms d (*i.e.*, $C\,D$ or $N\,O'$) remain unchanged, whether the load be inclined or vertical; and that is also the case with the intersection points i in the straight upper chord and with the lever arms n drawn from those intersection points to the centre lines of the web members prolonged. The equations established therefore for vertical loads may be used directly for inclined loads, it being necessary only to measure the lever arms as indicated in Fig. 2.

Inasmuch as any inclined load acting at G or K, Fig. 2, will not be carried directly to the abutment supports A and B, but will produce stresses, generally, in all members of the arch, the loads W_1 and W_6, acting as shown in Fig. 2, must be included,

among the moving loads acting on the structure, but the presence of those two loads does not change in any way the preceding equations; each such force must simply be given its proper lever arm.

As dead loads or own weight are always vertical, the stresses due to dead loading, as found in the preceding section of this article, will be used in connection with the stresses due to the inclined loads of this section.

Since all inclined loads have horizontal components, it is clear that the horizontal components H and H' of the two resultant reactions R and R' at A and B will not in general be equal for inclined loads, and they are not so shown in Fig. 2. The algebraic sum of the horizontal components of all the loads, however, must be equal and opposite to the sum of the horizontal components H and H' of the reactions.

The use of graphical methods in connection with inclined loading is easy and clear, as already indicated for vertical loadings, after the horizontal components H and H' of the reactions are known.

Greatest Stresses.

The greatest stresses in the members of a braced spandrel arch may be found by the ordinary influence-line method as applied to the two-hinged arch rib, or by finding the stresses in every member of the structure caused by an arbitrary unit moving load placed successively at each panel point. By tabulating results for all members of the arch, selection of stresses of the same sign for each member may be made so as to combine them into a maximum for each case.

CHAPTER XI.

CANTILEVERS.*

Art. 105.—Cantilever Structures—Positions of Loading for Greatest Stresses in the Cantilever Arm.

CANTILEVER structures are formed of continuous trusses for which the reactions and stresses are equally determinate with those in non-continuous trusses, because the points of contraflexure are fixed at the ends of the suspended span. Fig. 1 typifies a cantilever structure with an anchor arm at each end. The anchorage at A holds the anchor arm n in position under all conditions of loading, thus enabling the cantilever arms m to sustain the suspended span l. The latter is a simple, non-continuous truss with inclined end-posts, suspended at the lower extremities of the vertical tension members at C and D, thus readily allowing expansion and contraction to take place both in the suspended span itself and in the cantilever arms adjacent to it. As cantilevers are erected without lower false-works between the piers B and E, each cantilever arm, with the adjacent half of the suspended span, is built out from each of those main piers; hence the members $F\,G$ sustain compression during erection, while $H\,D$ take tension. These erection stresses are frequently very heavy, and the proper members must be designed to take them. At the same time, suitable details must be arranged so that after the structure is complete the requisite expansion and contraction can take place at each end of the suspended span. These results are usually accomplished by folding wedges, or other appliances, removed after erection, in connection with oblong pin-holes.

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* This chapter is devoted to the analytical treatment of cantilever structures. The graphical treatment may be found in "Influence Lines for Bridges and Roofs," by Wm. H. Burr and M. S. Falk.

Similarly, nearly the whole of the upper chord of the suspended span must be designed to take the tensile erection stresses, and nearly the whole of the lower chord the corresponding compression erection stresses. The erection stresses in the cantilever arms are, of course, the same in kind as those caused by the moving and fixed loads, and no reversion takes place.

The conditions of loading for the greatest stresses in the suspended span, in addition to the erection stresses, are precisely the same as those for ordinary non-continuous trusses, and they will receive no further attention here.

Greatest Web Stresses in Cantilever Arm.

In seeking the greatest web stresses in the cantilever arm, reference will be made to Fig. 1, and a perfectly general system of loading will be assumed. The chords of the cantilever arm

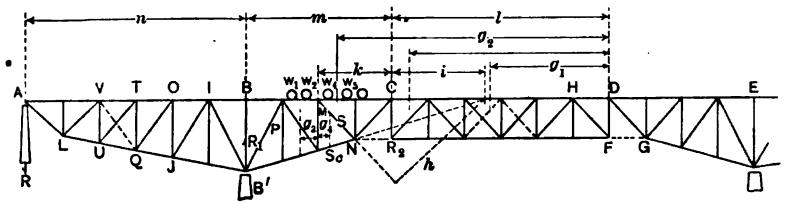

FIG. 1.

will also be assumed to be not parallel. Let the greatest stress S, in $M\,N$, be desired, and let the system of concentrated loads W_1, W_2, etc., extend over the entire suspended span l, and over the cantilever arm m, from C to some point to the left of M. Then let i represent the distance from C to the point of intersection of the chord sections in the same panel with $M\,N$, while h is the length of the normal dropped from the intersection point to $M\,N$ prolonged. Also let g_1 represent the distance from D to the centre of gravity of the load on l; g_2 the distance from the same point to the centre of gravity of the load to the left of C; g_3 the similar distance from M for the load to the left of M; and g_4 the similar distance from M of the loads W_4, W_5, etc.,

in the panel in question. The loads on the left of M are W_1, W_2, etc. The distance from C to M is represented by k. R is the downward reaction or upward pull on the anchorage at the extremity of n; while R_1 is the upward reaction at the main pier. The weights over the various portions of the structure will be represented as follows:

$\overset{l}{\Sigma} W$ for total load on l.
$\overset{m}{\Sigma} W$ " " " " m.
$\overset{k}{\Sigma} W$ " " " " k.
ΣW " " " " $m + l$.

The centre of gravity distances, g_1, g_2, g_3, and g_4 can be readily found by aid of tabulated moments or by any other convenient procedure.

Then, by moments in the ordinary manner:

$$R = \frac{m\, g_1}{n\, l} \overset{l}{\Sigma} W + \left(\frac{m + l - g_2}{n}\right) \overset{m}{\Sigma} W \quad . \quad . \quad . \quad . \quad 1.$$

$$R_2 = \textit{reaction at } C \textit{ from load on suspended span} = \frac{g_1}{l} \overset{l}{\Sigma} W \quad . \quad 2.$$

$$R_1 = R + R_2 + \overset{m}{\Sigma} W = \frac{g_1}{l} \left(\frac{m}{n} + 1\right) \overset{l}{\Sigma} W + \left(\frac{m + l - g_2}{n} + 1\right) \overset{m}{\Sigma} W \quad . \quad 3.$$

If the length of the panel in question is p, then that portion of the loads W_4, W_5, etc., resting in it and carried to M, is:

$$(W_4 + W_5 + \text{etc.}) \left(1 - \frac{g_4}{p}\right) . \quad . \quad . \quad . \quad . \quad 4.$$

If the cantilever arm be now supposed divided through the panel $M\,N$, and if the moments about the intersection of chords be taken of all the forces external to that portion of the structure to the left of this line of division, including the reactions R and $-\,R_1$, there will result:

$$S = \frac{1}{h}\left\{\frac{i\,g_1}{l}\overset{l}{\Sigma}W + (g_2 - l + i)\overset{m}{\Sigma}W\right.$$
$$- (W_1 + W_2 + \text{etc.} + W_4 + W_5 + \text{etc.})\,(k + i)$$
$$\left. - (W_1 + W_2 + \text{etc.})\,g_3 + (W_4 + W_5 + \text{etc.})\,(k+i)\,\frac{g_4}{p}\right\} \quad . \quad 5.$$

In order that S may be a maximum or a minimum, ΔS must be zero when g_1, g_2, g_3, and g_4 each varies by the same small amount, as when the loads are all advanced slightly to the left. It is to be remembered, however, that the variation of g_4 will be opposite in sign to that of the others. By thus making ΔS equal to zero, there will result as the desired condition:

$$W_1 + W_2 + \text{etc.} + (W_4 + W_5 + \text{etc.})\,\frac{(k+i)}{p} = \frac{i}{l}\overset{l}{\Sigma}W + \overset{m}{\Sigma}W.$$

$$\therefore (W_4 + W_5 + \text{etc.})\,\frac{(k+i)}{p} = \frac{i}{l}\overset{l}{\Sigma}W + \overset{k}{\Sigma}W. \quad . \quad . \quad 6.$$

In case there may be a number of maxima indicated by Eq. 6, the greatest must be sought by trial. Eq. 6 shows, as might have been anticipated, that the loads $W_1 + W_2 +$ etc., between the panel in question and the main pier, do not directly affect the conditions for greatest stress.

If the load is of the uniform intensity w, and x is the portion of the panel p covered by it, Eq. 6 is only satisfied by making $x = p$. This result might also have easily been anticipated, for it is clear that if the load is uniform it must at least reach to M, for the panel $M\,N$. The amount by which it may extend to the left of M is a matter of indifference.

If the chords are parallel, i is infinitely great, and Eq. 6 becomes:

$$W_4 + W_5 + \text{etc.} = \frac{p}{l}\overset{l}{\Sigma}W \quad . \quad . \quad . \quad . \quad . \quad 7.$$

In the general case, when Eq. 6 has fixed the proper position of loading, the corresponding values of g_1, g_2, g_3, and g_4 can be at once obtained by the methods already indicated, and Eq. 5 will then give the desired greatest value of S. In the case of a

uniform load, w, that equation takes a much simplified form, since:

$$W_1 + W_2 + \text{etc.} = 0; \qquad W_4 + W_5 + \text{etc.} = w\,p;$$

$$\overset{m}{\Sigma}\, W = w\,k; \qquad \overset{l}{\Sigma}\, W = w\,l;$$

$$g_1 = \frac{l}{2};\ g_2 = l + \frac{k}{2};\ g_3 = 0, \text{ and } g_4 = \frac{p}{2}.$$

Substituting these values in Eq. 5:

$$S = \frac{w}{2\,h}\left\{ i\,(l + k) + (k - p)\,(k + i) \right\} \quad . \quad . \quad . \quad 8.$$

If the chords are parallel and α is the angle between S and a vertical line, $h = (\infty = i)\cos\alpha$; hence:

$$S = w\left(\frac{l}{2} + k - \frac{p}{2}\right) \sec.\,\alpha. \quad . \quad . \quad . \quad . \quad . \quad 9.$$

With the same uniform load, w, per lineal foot, and with $k\,w$ on the cantilever arm m, Eqs. 1, 2, and 3 become:

$$R = w\frac{m}{n}\left(\frac{l}{2} + k - \frac{k^2}{2\,m}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 10.$$

$$R_2 = \frac{w\,l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 11.$$

$$R_1 = \frac{w\,l}{2}\left(\frac{m}{n} + 1\right) + w\,k\left(\frac{m}{n} - \frac{k}{2\,n} + 1\right) \quad . \quad . \quad . \quad 12.$$

If the load, w, covers the entire arm m, $k = m$, and:

$$R = w\frac{m}{2\,n}(l + m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 13.$$

$$R_2 = \frac{w\,l}{2}, \text{ and } R_1 = \frac{w\,l}{2}\left(\frac{m}{n} + 1\right) + w\,m\left(\frac{m}{2\,n} + 1\right) \quad 14.$$

If a single weight, W, rests on the suspended span, l, at the distance g_1 from D:

$$R = W\frac{m\,g_1}{n\,l};\ R_2 = \frac{W\,g_1}{l};\ \text{and } R_1 = \frac{W\,g_1}{l}\left(\frac{m}{n} + 1\right) \quad . \quad 15.$$

If a single weight, W, rests on the arm m at the distance k from C:

$$R = \frac{W(m-k)}{n}; \; R_2 = 0; \text{ and } R_1 = W\left(\frac{m-k}{n} + 1\right). \quad 16.$$

Greatest Chord Stresses in the Cantilever Arm.

Reference will again be made to Fig. 1 in which the compression web members are vertical; the notation will remain as before. The greatest chord stress, S_c, in any panel, as $M\,N$, will be sought. Let h^1 represent the normal dropped from M on the chord panel whose greatest stress is S_c; then, if moments be taken about M, h^1 will be the lever arm of S_c, and there will result:

$$S_c = \frac{1}{h^1}\left\{ R_1(m-k) - R(m+n-k) - (W_1+W_2+\text{etc.})\,g_3 \right\}. \quad 17.$$

$$\therefore S_c = \frac{1}{h^1}\left\{ g_2 \overset{m}{\Sigma} W - \frac{k\,g_1}{l} \overset{l}{\Sigma} W - (l+k) \overset{m}{\Sigma} W - (W_1+W_2+\text{etc.})\,g_3 \right\}. \quad 18.$$

For a maximum or minimum $\Delta S_c = 0$ when g_1, g_2, and g_3 are the only variables in the second member of the equation; hence:

$$\frac{k}{l} \overset{l}{\Sigma} W = \overset{m}{\Sigma} W - (W_1+W_2+\text{etc.}) = \overset{k}{\Sigma} W \quad . \quad . \quad 19.$$

This condition for a maximum is seen to be independent of i; hence it is precisely the same whether the chords are parallel or inclined. After the values of g_1, g_2, and g_3, corresponding to the position of loading fixed by Eq. 19, are inserted in Eq. 18, the desired chord stress will at once result. It is to be borne in mind, however, that there may be several maxima fixed by Eq. 19, and that the greatest of these, to be found by trial, is the "greatest stress" desired.

If the uniform load is w per lineal unit of structure, Eq. 19 is satisfied only by making $\overset{k}{\Sigma} W = k\,w$; hence, *the moving load must cover the suspended span, and the cantilever arm from its*

end to the vertical line through the centre of moments—i.e., the distance k. Eq. 18 then takes the value, for uniform loading:

$$S_c = -\frac{w\,k}{2\,h^1}(k+l) \quad . \quad . \quad . \quad . \quad . \quad . \quad 20.$$

For a panel in the horizontal chord, h^1 is simply the depth of truss at the origin of moments.

In case all web members are inclined, as indicated in Fig. 2,

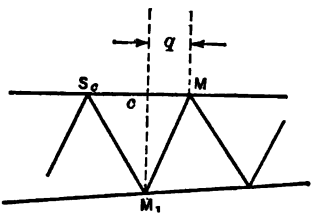

FIG. 2.

so that any lower chord panel point M_1 is at the horizontal distance q from that in the upper chord, Eq. 18 takes the form:

$$S_c = \frac{1}{h^1}\left\{ g_2 \overset{m}{\Sigma} W - \frac{(k+q)\,g_1}{l} \overset{l}{\Sigma} W - (l+k+q) \overset{m}{\Sigma} W \right.$$
$$\left. - (W_1 + W_2 + \text{etc.})\,(p-q)\,\frac{g_3}{p} \right\} \quad . \quad . \quad 21.$$

Again making $\Delta S_c = 0$, with g_1, g_2, and g_3 the only variables, there results for a maximum or minimum:

$$\overset{m}{\Sigma} W = \frac{k+q}{l} \overset{l}{\Sigma} W + (W_1 + W_2 + \text{etc.})\,\frac{p-q}{p} \quad . \quad . \quad 22.$$

Or,

$$\frac{k+q}{l} \overset{l}{\Sigma} W = \overset{m}{\Sigma} W - (W_1 + W_2 + \text{etc.})\,\frac{p-q}{p} \quad . \quad . \quad 23.$$

If $q = 0$, Eq. 19 at once results. The observations following Eq. 19, regarding i, g_1, g_2, and g_3 obviously hold true for this case also. If the moving load is of the uniform intensity w,

Eq. 23 is satisfied only by making it cover $q + p$; hence that equation takes the form:

$$k + q = k + \frac{p\,q}{p} = k + q; \quad . \quad . \quad . \quad . \quad . \quad 24.$$

which shows that *the uniform moving load must cover the suspended span, and the cantilever arm from its end to the further extremity of the panel in which the chord stress is sought.*

For this uniform moving load Eq. 21 takes the form:

$$S_c = \frac{w}{h^1}\left\{\frac{(k+p)^2}{2} - (k+q)\left(k+p+\frac{l}{2}\right) - (p-q)\frac{p}{2}\right\}. \quad 25.$$

If $q = 0$ in this equation, Eq. 20 immediately follows.

If the panel in the lower chord is under consideration, M, Fig. 2, becomes the origin of moments, and as the moving load is supposed to traverse the upper chord, the conditions for a maximum and the corresponding formulæ are precisely the same as for the case with vertical compression web members.

If, with all inclined web members, the moving load traverses the lower chord, the preceding conditions and formulæ apply precisely as they stand. It is only to be remembered that the formulæ involving q are to be used for the chord carrying the moving load.

If the tension members are vertical, and the compression members inclined, as in the Howe truss, or as exemplified by the inclined post, P, Fig. 1, the positions of moving load for greatest chord stresses will be at once found by making $q = p$ in Eq. 23, which gives:

$$\frac{k+p}{l}\overset{l}{\Sigma}W = \overset{m}{\Sigma}W \quad . \quad . \quad . \quad . \quad . \quad 26.$$

for the general load, and the same condition for a uniform load as that described in italics immediately after Eq. 24. By making $q = p$, in Eq. 25, the expression for the greatest chord stress under a uniform load, w, becomes:

$$S_c = -\frac{w}{h^1}\left\{\frac{(k+p)^2}{2} + \frac{(k+p)l}{2}\right\} = -\frac{(k+p)\,w}{2\,h^1}(k+p+l). \quad 27.$$

If the truss is of uniform depth, h^1 is that depth.

The preceding investigations hold true for any system of triangulation or for any system of loading. They do not apply to multiple systems of bracing, as such systems do not admit of exact calculation of stresses. When they are used, each system may be assumed to act independently of all the others (although the assumption cannot be definitely established) and to carry the system of greatest concentrations permitted by the moving load used, or else such a uniform load as will be practically equivalent to the real load.

Although there is not at the present time any constructive or other proper reason for the use of multiple systems of bracing in any pin structure of proportions hitherto employed, the above assumptions are within the limits of safety, and may be used where necessary.

Another constructive device is that shown in Fig. 3. It may be used advantageously in cases where the cantilevers

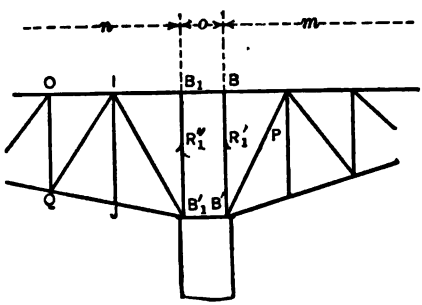

FIG. 3.

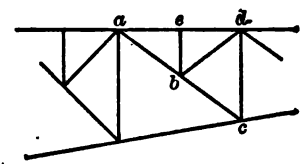

FIG. 4.

are supported on high iron or steel piers or towers. It leads to appreciable economy by the shortening of the clear cantilever opening, as well as the anchor arm. It consists in separating the feet of the two inclined posts, P and $I\,B'$, of Fig. 1, by some convenient distance, o, and omitting all bracing in the rectangle, $BB'B'_1B_1$, Fig. 3, so that no shear can be transferred past B' to the left, or past B'_1 to the right. Since this last condition exists, the reaction, R, and the positions of loading for all the maxima web and chord stresses in the cantilever arm, as given by the preceding formulæ, will hold for this case without any

change whatever. It is only to be carefully observed that m and n are to be measured from the extremities of the cantilever and anchor arms, respectively, to the extremities of the open panel, o, as shown in Fig. 3. The reaction, R'_1 at B', will equal the sum of the vertical components of all the inclined stresses in the first panel of m—*i.e.*, the total shear in that panel, added to the panel loads acting at B and B'. Similarly the reaction, R''_1 at B'_1, will equal the sum of the vertical components of all the inclined stresses in the first panel of n—*i.e.*, the total shear in that panel, added to the panel loads acting at B_1 and B'_1. Finally the reactions, R'_1 and R''_1, of Fig. 3, added together, will equal the reaction, R_1, of Fig. 1. Thus it is seen that the formulæ already established for the latter case will meet the former without any change whatever.

The case of the subdivided panel shown in Fig. 4 is also covered by the preceding formulæ.

Art. 106.—Positions of Moving Load for Greatest Stresses in the Anchor Arm, and the Greatest Stresses Themselves.

No load placed on the anchor arm of a cantilever structure will affect in any way the reaction or any shear in the cantilever arm, except such as may result from the slight deflectional movement of the latter, and that influence is too small to be expressed in appreciable amounts. It at once follows from this fact that all stresses in the anchor arm, due to loads on that arm, may be computed precisely as if it ($A\ I\ B'\ L$ of Fig. 1) were a simple, non-continuous truss supported at each end (*i.e.*, A and B' of Fig. 1). The greatest of these stresses so found are then to be combined with the greatest stresses of the same kind due to the load on the cantilever arm and resulting from the reaction, R, at the anchorage; this combination will give the resultant greatest stresses desired.

Although the preceding general observations are comprehensive and sufficient for the determination of all the greatest stresses desired, it will be well to extend them in some detail, without, however, establishing any formulæ.

Inasmuch as all upper chord stresses in the anchor arm, due to load on that arm only, will be compressive, it is clear that the greatest tension in the upper chord, so far as it may exist, will be found with no moving load on the anchor arm, and with the moving load so placed on the cantilever arm as to give the maximum bending moment (and, hence, chord stresses) over the main pier between the anchor and cantilever arms. This position of the moving load on the cantilever arm and the corresponding greatest chord stresses can be found for the various cases from Eqs. 18, 19, 20, 21, 23, and 25 of the preceding article. The resulting moment over the main pier, divided by n, will give the reaction, R, with which the greatest tension throughout the upper chord and the greatest compression throughout the lower chord are to be found—*i. e.*, so far as this tension or compression exists.

If the anchor arm is long in comparison with the cantilever arm, the fixed load compression in the upper chord and tension in the lower may be so great that there will be but little upper chord tension and lower chord compression, even for the conditions producing maxima.

Inasmuch as all loads on the suspended span and cantilever arms produce tension in the upper chords of the anchor arms and compression in the lower, the positions of moving load for the greatest compressions in the upper chord of either anchor arm, or greatest tensions in the lower, are precisely the same as if it (the anchor arm) were a simple, non-continuous truss, the cantilever arms and suspended span, at the same time, carrying no moving load. The greatest compressive stresses in the upper chord and tensile stresses in the lower chord thus found in the anchor arm, as for a simple, non-continuous span, are to be combined with the upper chord tension and lower chord compression produced by the fixed load of the cantilever arms and suspended span, whence will result the maxima chord stresses desired.

The greatest shears in the anchor arm will evidently be affected by the magnitude of the reaction R, Fig. 1, at its extremity. If a downward shear, producing tension in members

sloping similarly to $O\,Q$, or compression in $I\,B'$, is called a main shear, then any condition of loading on the structure which will increase the downward reaction, R, will increase the main shears and, hence, the stresses in the main web members, as $A\,L$, $L\,V$, $U\,T$, etc., and $I\,B'$ will be called. But the maximum value of the downward moving load reaction or pull at the extremity of the anchor arm will be found with no moving load on the latter, and for the maximum value of the bending moment over the main pier B', as given by Eqs. 18, 20, 21, and 25, of the preceding article, as already stated on page 387. Finally, that part of the maximum main shear in any panel of the anchor arm due to the moving load on that portion of the structure alone, is found precisely as if it were a simple, non-continuous truss with the given system of loading, advancing from the extremity of the anchor arm towards the main pier. The positions of moving load, therefore, for the main shears or main web stresses are, first, so much on the cantilever arms and suspended span as will produce the maximum downward pull at the extremity of the anchor arm, and, second, with this constant condition in the main span of the structure, an advancing moving load from the extremity of the anchor arm, treated as a simple, non-continuous span, to the main pier. The maximum moving load shears or stresses thus found, combined with the fixed-load shears or stresses in the anchor arm, will give the resultant greatest web member stresses desired.

It may sometimes happen that the anchor arm will be so long that the greatest downward pull at its extremity due to the moving load will be less than an existing upward reaction due to the fixed load. In such a case no anchorage, of course, will be required, as the reaction at the extremity of the anchor arm will always be upward. The preceding conditions for greatest main web stresses, however, hold in all cases without exception.

The greatest counter shears, and, hence, stresses, in counter web members such as $Q\,V$, or $V\,L$ if under compression, or $A\,L$ in tension, Fig. 1, will, for the same general reasons just stated in connection with the greatest main shears, be found

under the moving load advancing from the main pier B' to the extremity of the anchor arm, under the supposition that the latter is a simple, non-continuous truss, with no moving load whatever on the cantilever arms or suspended span.

If a portion of the web members are vertical posts, as shown in Fig. 1, so that the kind of stress in them is the same whether acting as counters or main web members, those nearest the extremity of the anchor arm will probably take their greatest stresses as counters, but those more remote are doubtful and must be computed both as main and counter members in order to find the greatest values. Those near the main pier will take their greatest stresses as main web members.

Art. 107.—Positions of Moving Load for Greatest Stresses in Anchor Spans and the Greatest Stresses Themselves.

The anchor span is shown as the span n in Fig. 5, on the left of which there is supposed to be a cantilever arm similar to m on the right. The anchor span is always of such length and weight that the reactions, R and R_1, are invariably upward. Hence the anchor span is identical in character with the anchor arm, whose length and weight are so great that the reaction at its extremity over the anchorage is always upward. Hence the positions of moving load for the greatest stresses are precisely the same as those established for the anchor arm in the last article.

The greatest moving-load stresses in the span n, Fig. 5, are to be determined precisely as if the trusses were non-continuous;

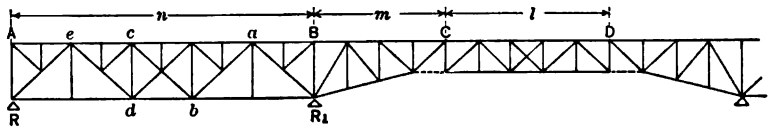

FIG. 5.

and the fixed-load stresses are to be found in the same trusses under exactly the same conditions. The greatest difference of moments at the two sections, $B\,R_1$ and $A\,R$, is then to be found

and divided by the span length n, *thus giving a reaction which is to be treated precisely as the downward reaction at the extremity of the anchor arm in the preceding article.* This downward reaction will produce stresses in the chord and web members of the anchor span, which are next to be determined in the usual manner for a load hanging at the extremity of the span n, with the other extremity fixedly held. If both cantilever arms flanking the span n are of the same length m, and of the same weight similarly disposed, the fixed-load moments at the sections $A\ R$ and $B\ R_1$ will always be equal (but not otherwise), and the difference in moments mentioned above will be entirely due to the moving load on one of the adjoining cantilever spans. Any difference of moments at those two sections resulting from dissimilarity or inequality of fixed loads must be added to or subtracted from the unbalanced moving-load moment just described. Finally, the fixed-load balanced moment at either of the sections $A\ R$ or $B\ R_1$ is then to be determined and divided by the depth of the truss in order to find the uniform tension throughout the upper chord, and the uniform compression throughout the lower, if the truss has a uniform depth. The stresses resulting from these four sources—viz., the moving load and the fixed load on the span n; the unbalanced moment over the piers due to possibly both moving and fixed loads, and the balanced moment due to fixed load only—are to be so combined as to produce maxima in each and all the truss members.

The greatest web stresses will require the greatest unbalanced moving-load moment alternately at each end of the anchor span, with the usual progressive movement of moving load for ordinary non-continuous trusses; but the greatest upper chord compression and lower chord tension will be found with no moving load on the adjacent cantilever arms or suspended spans.

The greatest upper chord tension and lower chord compression will be found by combining the fixed-load stresses, found as indicated above, with the moving-load stresses induced by the greatest possible moving-load moments simultaneously at each end of the anchor span, the latter, at the same time, being entirely free from moving load.

If the anchor span is not of uniform depth, the balanced fixed-load moments at its extremities will induce stresses in the web members, which are to be determined in addition to those in the chords for combination with the other stresses described above in the search for the maxima.

Art. 108.—Wind Stresses.

The stresses due to the action of the wind on the various parts of a cantilever structure are important in its design. The conditions of wind loading are much more varied than in ordinary non-continuous spans, and the resulting stresses must be computed with care and thoroughness. The wind pressures against the different members of the structure constitute a fixed load, and all resulting stresses are to be found by precisely the same general methods as given in the preceding articles for the vertical fixed loads—*i. e.*, own weights.

Fig. 6 shows the horizontal system of lateral bracing between the horizontal upper chords, *A B C D E*, of Fig. 1. If the wind

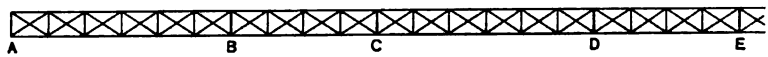

FIG. 6.

blows on the entire structure in a given direction, as shown by the arrow in Fig. 6, there will in general be a horizontal reaction induced at *A*, in direction and amount depending on the relative proportions of anchor and cantilever arms and suspended truss. This reaction must be provided for, at *A*, by a proper connection between the end of the anchor arm and masonry at that point, so designed that the requisite longitudinal thermal expansion and contraction may at the same time take place. The amount of this reaction is to be determined precisely as the fixed-weight reaction, *R*, at the same point, in Fig. 1. This reaction, *R*, will be composed of two parts, one of which is due to the wind pressures along the upper chord, *A B C D*, Fig. 1, and the other to those along the lower chord, *A L B′ N G*; and each is to be found as indicated above. By means of these horizontal re-

actions at A, and the panel wind pressures at the upper and lower chord points, all the fixed-load wind stresses in the upper and lower lateral systems, both of which are projected in Fig. 6, may be at once completely established.

The wind pressures against the moving train form a moving load in those chords carrying the train (the upper chords in Fig. 1, and the intermediate and lower chords in Fig. 7). The conditions under which these moving wind loads produce their horizontal reactions at A, and their greatest stresses in the web and chord members of the lateral system, are precisely the same as those described in detail for the vertical or train-moving loads in the preceding articles. The cantilever conditions, so to speak, including span lengths for the former loads, are identical with those for the latter. Hence the positions for the wind maxima must be determined by exactly the same general methods as those used for the vertical loads. The computations for the moving wind stresses are simplified by the fact that that load is of uniform intensity.

The combination of the fixed and moving wind load stresses according to the usual methods will give the desired resultants in all the members.

In the case of the anchor span of Fig. 5, the same general observations as those already made are to be applied. The fixed

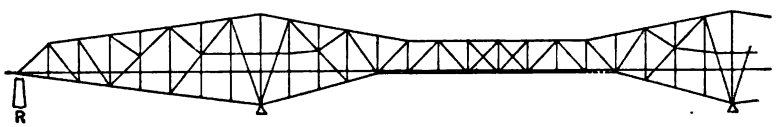

FIG. 7.

wind loads on the span are to be treated precisely like the fixed vertical loads, and the resulting wind stresses in the upper and lower lateral systems are to be computed accordingly. Again, the moving wind loads accompany the moving vertical or train loads, and all the conditions determining the greatest stresses for the latter determine those for the former also. The same set of greatest moving wind stresses must, therefore, be found

as for the moving vertical loads, and followed by their combination with the fixed wind stresses, in order to reach the resultant wind stresses desired.

Inasmuch as it is known that the highest wind pressures cover comparatively small areas, it will usually be necessary to supplement the preceding wind computations, which are based on the assumption that the wind presses equally over the entire structure, by others resulting from the application of the highest pressure to some particular portion, as the clear cantilever span or the anchor span, or, possibly, the anchor arm, and provide for the resulting stresses.

The extent to which these supplementary and special computations are to be made will depend upon the judgment of the engineer acting on the circumstances of each case.

The transverse bracing in the vertical and inclined planes of the various pairs of web members is to be designed under the same assumptions of wind transference, and in accordance with the same principles used for the same general purposes in ordinary non-continuous spans. The wind loads in the upper chords of the suspended trusses of the cantilever span should always be carried down to the lower chords of the cantilever arms at their extremities, and along those chords to the piers, in order that the overturning wind effect on the cantilever and anchor arms may be reduced to a minimum. In the same manner the wind loads in the upper chords of the cantilever and anchor arms, which are resisted by the main piers, should be carried down to them by the transverse bracing between the inclined posts acting as portals at those places. In high cantilever trusses the overturning wind effect is frequently a serious matter, and it should be carefully computed in all cases. It may transfer a considerable portion of the fixed load in the windward truss into the leeward. It may also considerably increase the downward pull on the windward side of the anchorage, and relieve that at the leeward side by the same amount. These changes in the downward pull on one side of the anchorage can be found by dividing the wind moment about the top of the pier between the anchor and cantilever arms by the transverse width between

the trusses where they are attached to the anchorage. The same effects occur, of course, similarly in the anchor span. These overturning effects, resulting from the pressure of the wind against the structure, affect the fixed-load stresses only. It may also be necessary to consider the truss stresses arising from the overturning effect of the wind on the train. This, however, will depend upon the special circumstances affecting any particular case.

Art. 109.—Economic Lengths of Spans and Arms.

The problem of the relative lengths of suspended spans, cantilever arms, anchor arms, and anchor spans for the most economical amounts of material in them is not capable of exact mathematical treatment, but may be simply solved in a manner sufficiently accurate for all practical purposes. In order to do this, it is only necessary to observe that the weight per lineal foot of any single span of bridge structure, omitting the floor, is very closely found by multiplying the span length by a numerical factor, determined by actual computation of weights of similar spans. In the case of a cantilever arm, the length of that arm would be used instead of span length. In the analysis which follows there will be used:

a = cantilever arm factor; b = suspended span factor;
c = anchor arm factor; e = anchor span factor;
L = length of cantilever span = $i + 2m$ of Fig. 1, Art. 105;
l = length of suspended span; n = length of anchor arm.

*Economic Lengths of Suspended Span, Cantilever Arm, and Anchor Arm.**

The total length of the entire structure, including the two anchor arms, supposed to be invariable in this analysis, is:

$$S = l + 2m + 2n \quad . \quad . \quad . \quad . \quad . \quad 28.$$

* Acknowledgment is due Prof. D. B. Steinman for improving the author's original solution of this problem.

The total weight of the entire structure of two anchor spans and the main or cantilever span is:

$$W = b\,l^2 + 2\,a\,m^2 + 2\,c\,n^2 \quad . \quad . \quad . \quad . \quad 29.$$

As $d\,S = 0$, Eq. 28 gives:

$$d\,l = -\,2\,d\,m - 2\,d\,n.$$

The desired values of l, m, and n, considered as variables, must be such as to make W a minimum. Hence, differentiating W and using the value of $d\,l$ given above:

$$d\,W = (4\,a\,m - 4\,b\,l)\,d\,m + (4\,c\,n - 4\,b\,l)\,d\,n. \quad . \quad 30.$$

In order that W shall be a minimum:

$$\left.\begin{aligned} \frac{d\,W}{d\,m} &= 0 = 4\,a\,m - 4\,b\,l \\ \frac{d\,W}{d\,n} &= 0 = 4\,c\,n - 4\,b\,l \end{aligned}\right\} \therefore b\,l = a\,m = c\,n \quad . \quad . \quad 31.$$

As $b\,l$, $a\,m$, and $c\,n$ are weights per linear unit of the trusses of the three principal parts of the structure, Eq. 31 shows that when the lengths of these parts are so chosen as to attain the greatest economy of material, the weight of trusses per linear unit is the same throughout the entire structure.

As $L = l + 2\,m$, and using the value of l given by the first two members of Eq. 31:

$$\text{Cantilever arm} = m = \frac{b}{a + 2\,b}\,L \quad . \quad . \quad . \quad . \quad 32.$$

Similarly, by taking the value of m from the same source:

$$\text{Suspended span} = l = \frac{a}{a + 2\,b}\,L \quad . \quad . \quad . \quad . \quad 33.$$

Using Eqs. 31 and 33:

$$\text{Anchor arm} = n = \frac{b}{c}\,l = \frac{a\,b}{c\,(a + 2\,b)}\,L \quad . \quad . \quad . \quad 34.$$

It is obvious that l, m, and n may be expressed in terms of any quantity proportionate to L.

For lengths of cantilever openings not great, there may be taken approximately $a = 40$, $b = 14$. These values inserted in Eqs. 1 and 2 will give nearly $l = .6\,L$ and $m = .2\,L$. It has been found in actual design, however, that l should not exceed about $.5\,L$ to $.55\,L$ with m about $.25\,L$. When the length of cantilever opening becomes large, perhaps 1,500 to 1,800 ft., it is advisable to take the length of suspended span, not more than about $.4\,L$. In other words, as the length of cantilever opening becomes large, the ratio of the length of suspended span divided by the length of the cantilever opening becomes proportionately less.

In this approximate analysis, it should be stated that some of the earlier short-span cantilever structures have been built in which the weight factors a and b were about half the values taken above. They were mostly single-track railroad bridges with far lighter loads than are now (1913) used.

Approximately, and corresponding to the values of a and b used above, there may be taken $c = 20$.

Anchor Span.

The case of an anchor span, Fig. 5, page 389, is covered by Eq. 34; in fact, an anchor span is analytically the same as two superposed anchor arms.

If Eqs. 28 and 29 be written with n in the place of $2\,n$, it will be found that Eqs. 31 and 34 will result as would be anticipated. The weight factor e, however, must be taken larger than for the anchor arm. Approximately, as an illustration:

$$\text{For anchor arm, } c = 20 \text{ and } \frac{a\,b}{c\,(a + 2\,b)} = .412 \quad . \quad . \quad 35.$$

$$\text{For anchor span, } e = 25 \text{ and } \frac{a\,b}{e\,(a + 2\,b)} = .33 \quad . \quad . \quad 36.$$

The reversal of stresses and the heavy bending moments found in the anchor span make its weight per linear unit relatively

high and its length relatively short. Wherever navigation interests are important they will usually require a materially longer span than indicated by Eq. 36.

Art. 110.—Width Between Centres of Cantilever Trusses.

The least permissible width between centres of cantilever trusses may readily be determined with sufficiently close approximation for such a purpose.

It has been found by experience that under good design a satisfactory degree of lateral stability for an ordinary simple truss bridge may be attained by making the width between centres of trusses at least $\frac{1}{18}$th of the span length. Hence, it may be stated that the width between the centres of cantilever trusses must at least be equal to the corresponding width of a simple ordinary truss bridge giving the same deflection as the cantilever and with trusses separated $\frac{1}{18}$th of the span length.

Let p be the uniform horizontal lateral (wind) pressure against each structure per linear foot. Also let I be the moment of inertia of the normal section of each structure about a vertical axis midway between the trusses, supposed to be the same for the two structures and constant throughout each span.

Then if u is the centre deflection in each case, and if E is the modulus of elasticity, remembering that the length of the cantilever span is $l + 2m$, the centre deflection of the cantilever trusses will be:

$$u = \frac{p}{EI}\left\{\frac{5\,l^4}{384} + \frac{l}{2}\frac{m^3}{3} + \frac{m^4}{8}\right\} \quad . \quad . \quad . \quad . \quad 37.$$

Hence for $l = .5\,L$, and $m = .25\,L$:

$$u = .0026\frac{p\,L^4}{EI} \quad . \quad . \quad . \quad . \quad . \quad 38.$$

And for $l = .4\,L$, and $m = .3\,L$:

$$u = .00315\frac{p\,L^4}{EI} \quad . \quad . \quad . \quad . \quad . \quad 39.$$

The same deflection of a simple truss of span l^1 and uniformly loaded is·

$$u = \frac{5}{384}\frac{p l^{1^4}}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 40.$$

Equating the second members of Eqs. 38 and 40, and then the second members of Eqs. 39 and 40:

$$\left.\begin{aligned} &\text{For } l = .5\,L;\ l^1 = .668\,L \\ &\text{For } l = .4\,L;\ l^1 = .701\,L \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad 41.$$

If d is the width between trusses:

$$\text{For } l = .5\,L;\ d = \frac{l^1}{18} = \frac{L}{27} \quad . \quad . \quad . \quad . \quad 42.$$

$$\text{For } l = .4\,L;\ d = \frac{l^1}{18} = \frac{L}{25.7} \quad . \quad . \quad . \quad . \quad 43.$$

The trusses for a cantilever span, therefore, should be placed a distance from centre to centre not less than about $\frac{1}{25}$th to $\frac{1}{27}$th of the length of the span.

This approximate solution of the problem is in substantial accord with the best practice of cantilever construction.

Art. 111.—The Cantilever Without Suspended Span.

A cantilever structure comprising two anchor arms, two cantilever spans, and an anchor span between the two latter, like the Blackwell's Island Bridge, New York City, becomes continuous from anchorage to anchorage when the suspended spans are displaced by rocker arms, vertical in the present case, connecting the upper and lower extremities of two adjacent cantilever arms. The stresses in the different members of the trusses cannot, therefore, be determined by the ordinary procedures applicable to statically determinate structures. It is necessary first to find the stresses (reactions or shears) in the rocker arms by the aid of formulæ involving the elastic properties of the materials of the structure, the latter being loaded in any manner whatever. The continuity of the trusses makes it

necessary to recognize, in the analysis, the simultaneous existence of the two rocker arm reactions or shears.

The complete determination of the general values of those reactions or shears is as follows:

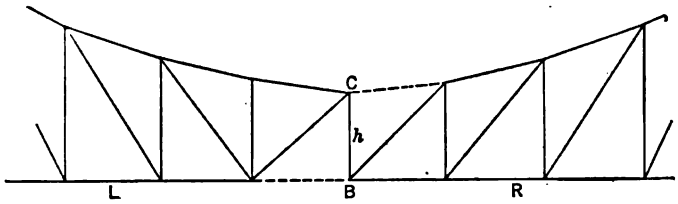

FIG. 8.

The Two Reactions or Shears R_3 and R_4.

To make the case general, it will be assumed that the lengths of cantilever arms on the right and left, respectively, are unequal and represented by l_R and l_L. The length of the rocker arm

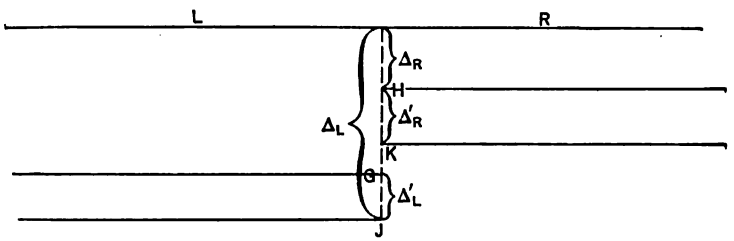

FIG. 9.

$C\,B$ is h, and its area of cross-section A. The line $L\,R$ in Fig. 9 is supposed to represent the lines of the cantilever arms, at the same elevation, before the structure is so loaded as to cause unequal deflections.

The principal quantity to be found is the force or reaction exerted between the ends of the cantilevers through the rocker arm $B\,C$ and when loaded in any arbitrary manner. All other desired quantities can then be at once determined.

If all connection between the extremities of the cantilever arms be cut when those arms are loaded in any arbitrary manner, in general the end of the right arm will have the deflection Δ_R,

while the deflection of the left end will be Δ_L. If, on the other hand, the ends are connected by an elastic member they will deflect to some point G and K between the lower limits of Δ_R and Δ_L. Let the force or reaction exerted in this elastic connection be represented by R_2. Also in Fig. 9 let Δ_R^1 represent $H K$, while Δ_L^1 represents $J G$, remembering that $H K$ and $J G$ are the downward and upward deflections produced by the force or reaction R_2 acting at the end of each cantilever arm as if the latter were free of any end connection. Then let ω be any arbitrary load supposed to act at the extremity of each free cantilever arm and to produce there the deflections δ_R and δ_L in the right and left arms respectively. Since these deflections are proportional to the forces producing them:

$$\frac{R_2}{\omega} = \frac{\Delta_R^1}{\delta_R} \text{ and } \frac{R_2}{\omega} = \frac{\Delta_L^1}{\delta_L} \quad . \quad . \quad . \quad . \quad . \quad 44.$$

The change of length of the rocker arm $C B$ caused by the stress in it is: $\frac{R_2 h}{E A}$; E being the coefficient of elasticity.

In this analysis downward deflection will be considered positive and upward deflection negative. In the case of Fig. 9, Δ_L^1 would be negative.

Furthermore:

$$\Delta_R^1 - \Delta_L^1 = \Delta_L - \Delta_R - \frac{R_2 h}{E A} \quad . \quad . \quad . \quad . \quad . \quad 45.$$

Substituting for Δ_R^1 and Δ_L^1 from Eq. 44 in Eq. 45:

$$R_2 = \frac{(\Delta_L - \Delta_R)\,\omega}{\delta_R - \delta_L}\left[\frac{1}{1 + \frac{h\,\omega}{E A\,(\delta_R - \delta_L)}}\right] \quad . \quad . \quad . \quad 46.$$

If Fig. 9 be supposed applied to span l_4, Δ_R will be Δ_L of Eq. 46, while Δ_L will be Δ_R of that equation in consequence of the symmetry of arrangement of spans l_2, l_3, and l_4.

If the division indicated in the bracket of the second member of Eq. 46 be carried out for two terms only, there may be written approximately:

$$R_2 = \frac{(\Delta_L - \Delta_R)\,\omega}{\delta_R - \delta_L}\left[1 - \frac{h\,\omega}{EA\,(\delta_R - \delta_L)}\right]. \quad . \quad . \quad 47.$$

There is no special advantage in the form of Eq. 47. Eq. 46 may be as easily used in all computations.

The fraction whose numerator is $h\,\omega$ in the bracket of the second members of Eqs. 46 and 47 is the allowance or correction to be made in the value of the reaction R_2 for the extension or compression of the rocker arm, and it may be omitted if that member is very short and of large cross-sectional area.

Special Values of Reaction R_2.

For the purposes of this analysis an upward reaction will be considered negative and a downward reaction positive.

If the load producing R_2 is wholly on the left cantilever arm, so that $\Delta_R = 0$, Δ_L only existing, Eq. 46 at once gives:

$$R_2^{\mathrm{I}} = \frac{\Delta_L\,\omega}{\delta_R - \delta_L}\left[\frac{1}{1 + \dfrac{h\,\omega}{EA\,(\delta_R - \delta_L)}}\right]. \quad . \quad . \quad 48.$$

On the other hand, when the right arm is loaded so that $\Delta_L = 0$, Δ_R only existing, Eq. 46 will give:

$$R_2'' = -\frac{\Delta_R\,\omega}{\delta_R - \delta_L}\left[\frac{1}{1 + \dfrac{h\,\omega}{EA\,(\delta_R - \delta_L)}}\right]. \quad . \quad . \quad 49.$$

It is thus disclosed that the general value of the reaction R_2 is the algebraic sum of the two special reactions. Hence, if a certain loading produces a special reaction R_2^{I} and another loading the special reaction R_2'', each causing a corresponding set of stresses in each arm, the resultant reaction caused by the concurrent action of the two loadings will be the sum of the two special reactions, and the stresses produced in the various truss members will be the algebraic sum of the stresses corresponding to those two special reactions.

*Application to Five-Span Bridge.**

All the preceding formulæ are perfectly general and applicable to any number of spans. In applying them to any special case, as that indicated by Fig. 10, due regard must be shown to the values and signs which Δ_L and Δ_R will then take. This may be illustrated by considering the particular structure shown by Fig. 10. There can be no shear or reaction R_2 in the span l_2, Fig. 10, without a corresponding shear or reaction R_4 in the span l_4; and the same observation is to be made regarding the deflections at those two points.

If u_2 is the deflection at C, Fig. 10, due to any arbitrary load, ω (as 1 or 1,000 pounds), acting at D in l_4, then the deflection

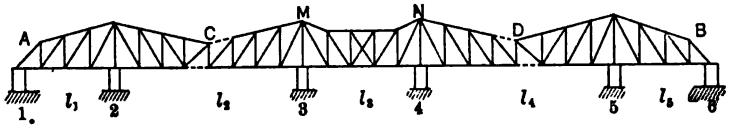

FIG. 10.

at the same point C due to the shear or reaction R_4 at D, expressed in the same unit ω, will be:

$$u_2 \frac{R_4'}{\omega}.$$

Hence in Eq. 46, there must be written for Δ_R the two terms $\Delta_R + \frac{u_2 R_4}{\omega}$; Δ_R now representing the deflection of the extremity of the cantilever $C\,M$ due to any loads resting upon the structure between C and D. Similarly if d_4 represents the deflection of the extremity of the cantilever $D\,N$ due to the action of the same arbitrary load or unit at C, in the value of R_4:

$$R_4 = \frac{(\Delta_R - \Delta_L)}{\delta_L - \delta_R}\left[\frac{\omega}{1 + \frac{h\,\omega}{E\,A\,(\delta_L - \delta_R)}}\right] \quad . \quad . \quad . \quad 50.$$

* The "Five-Span Bridge" is the Queensboro Bridge across the East River, at New York, on the design and carrying capacity of which the author, as consulting engineer, made an extended report in 1908.

there must be written $\Delta_L + \frac{R_2 u_4}{\omega}$ for Δ_L in Eq. 50; Δ_L now representing deflection due to load on $C D$.

There may then be written the following two equations by the aid of Eqs. 46 and 50.

$$R_2 = \frac{\left(\Delta_L - \Delta_R - \frac{u_2 R_4}{\omega}\right)}{\delta_R - \delta_L} \left[\frac{\omega}{1 + \frac{h\omega}{EA(\delta_R - \delta_L)}}\right] \quad . \quad . \quad 51.$$

$$R_4 = \frac{\left(\Delta_R - \Delta_L - \frac{R_2 u_4}{\omega}\right)}{\delta_L - \delta_R} \left[\frac{\omega}{1 + \frac{h\omega}{EA(\delta_L - \delta_R)}}\right] \quad . \quad . \quad 52.$$

It must be carefully borne in mind that the same letter in the two Eqs. 51 and 52 does not in general possess the same numerical value for any given structure, although in some cases h and A may have the same values.

Let there be placed in Eq. 51:

$$a_2 = \frac{\omega}{(\delta_R - \delta_L) + \frac{h\omega}{EA}}, \quad . \quad . \quad . \quad . \quad . \quad 53.$$

also, in Eq. 52:

$$a_4 = \frac{\omega}{(\delta_L - \delta_R) + \frac{h\omega}{EA}}; \quad . \quad . \quad . \quad . \quad . \quad 54.$$

then:

$$R_2 = a_2(\Delta_L - \Delta_R)_2 - \frac{a_2 u_2 R_4}{\omega} \quad . \quad . \quad . \quad . \quad . \quad 55.$$

$$R_4 = -a_4(\Delta_L - \Delta_R)_4 - \frac{a_4 u_4 R_2}{\omega} \quad . \quad . \quad . \quad . \quad 56.$$

Eliminating between Eqs. 55 and 56:

$$R_2 = a_2 \frac{(\Delta_L - \Delta_R)_2 + \frac{a_4 u_2}{\omega}(\Delta_L - \Delta_R)_4}{1 - a_2 a_4 \frac{u_2 u_4}{\omega^2}} \quad . \quad . \quad . \quad 57.$$

$$R_4 = -a_4 \frac{(\Delta_L - \Delta_R)_4 + \frac{a_2 u_4}{\omega}(\Delta_L - \Delta_R)_2}{1 - a_2 a_4 \frac{u_2 u_4}{\omega^2}} \quad . \quad . \quad . \quad 58.$$

The subscripts 2 and 4 affecting the parentheses in the numerators of the second member of Eqs. 57 and 58 indicate that the quantities $(\Delta_L - \Delta_R)$ belong to the spans l_2 and l_4 respectively.

Eqs. 57 and 58 give the two shears or reactions R_2 and R_4 at the extremities of the cantilever arms. When these two reactions are determined, the ordinary analytic or graphic methods for determining the stresses in all members of the entire structure from anchorage to anchorage may be applied.

It will be observed that in the second members of Eqs. 57 and 58, the only variable quantities are Δ_R and Δ_L, which are to be found by the aid of stresses determined by ordinary statical methods.

The preceding formulæ were used in investigating the design and carrying capacity of the Queensboro Bridge at New York City.

It is obvious that a cantilever structure without a suspended span is a continuous structure with but a single point of contraflexure in the main span fixed at the centre of the latter. It is difficult to imagine conditions which may justify the omission of the suspended span.

APPENDIX I.

Limiting Spans and Depths of Stiffening-Trusses.

It is a matter of importance to determine, approximately at least, the economic limits of span for those structures adapted to long-span construction. Prof. D. B. Steinman, in his valuable little book, "Suspension Bridges and Cantilevers," and in two papers presented before the Congress of Engineering and Scientific Societies at Spokane, Washington, 1913, has presented some effective comparisons based probably upon the most extended investigations in comparative designs yet made. In order to canvass the problems involved in a specific manner, he classified the greatest spans practicable for suspension and cantilever bridges as follows:

1. The limiting spans or the greatest spans which may be designed.

2. The maximum spans or the greatest spans which can be built.

3. The limiting economic spans or the greatest spans which it would pay to build.

4. The span of equal cost for the two types; in other words, the span at which the cantilever ceases to be economically superior to the suspension bridge.

Without going into details of the investigations made by Professor Steinman, it should be stated that he prepared complete designs and estimates of three suspension bridges of 1,500 feet, 2,250 feet, and 3,000 feet spans, and of three cantilevers of 1,000 feet, 1,500 feet, and 2,000 feet spans, respectively. He assumed the bridges to be designed to carry combined railway, highway, and sidewalk traffic, inasmuch as the widths of these structures would obviously be great enough to accommodate such traffic. Structures so costly would certainly be built only in localities where the total volume of traffic would be

proportionately great. The total maximum load in Professor Steinman's estimates was taken at 18,000 pounds per linear foot, corresponding to that used in at least one of the large bridges across the East River at New York City. He furthermore assumed wire cables for the suspension bridges, each with a diameter of twenty-four inches. The diameters of the wire cables of the suspension bridges across the East River at New York are fifteen and five-eighth inches for the oldest structure, eighteen and three-quarter inches for the Williamsburg Bridge, and twenty-one and a half inches for the Manhattan Bridge. The results of Professor Steinman's investigations may be expressed in tabular form as follows:

	Suspension Bridge	Cantilever Bridge
1. Greatest span limit	14,700 ft.	5,600 ft.
2. Greatest practicable span	4,900 "	3,060 "
3. Greatest economic span	3,170 "	2,700 "
Span of equal cost	1,670 ft.	

It may be stated that for bridges designed to carry heavy traffic the upper advantageous limit of the cantilever span will be reached with materials available at the present time (1913) at some point between about 1,500 feet and 1,700 feet, and that the lower advantageous limit of span for the stiffened suspension bridge can be taken at the same point.

Obviously, for classes of light traffic where the platform carrying it need not be much stiffened, the application of the suspension-bridge cable has no lower limit, as is attested by many simple or even crude constructions.

The most advantageous depth of stiffening-truss for the stiffened suspension bridge may be taken approximately at one-fortieth to one-sixtieth of the span length.

For a valuable and interesting analytic treatment of these questions, Professor Steinman's book and papers may be consulted.

APPENDIX II.

FORMULÆ FOR REINFORCED CONCRETE.

Art. 1.—Notation and General Formulæ.

THE principal formulæ required in the treatment of reinforced concrete beams of rectangular cross-section are simple variants of the formulæ of the common theory of flexure, and they may easily be written by the aid of the following notation:

E_2 = coefficient of elasticity of the steel.

E_1 = coefficient of elasticity of the concrete in compression or tension.

A_1 and A_2 are the areas of normal section of the concrete and steel, respectively.

I_1 and I_2 are the moments of inertia of A_1 and A_2, respectively, about the neutral axis of the normal section.

k_1 = greatest intensity of bending compression in the concrete.

k' = greatest intensity of bending tension in the concrete.

c = greatest intensity of bending compression in the steel.

t = greatest intensity of bending tension in the steel.

b = breadth of the concrete.

h_1 and h_2' are total depths of the concrete and steel, respectively.

h_2 = vertical distance between the centres of the steel reinforcing members.

d_1 = distance of extreme compression "fibre" of the concrete from the neutral axis.

d_2 = distance of the centre of the compression steel reinforcing member from the neutral axis.

d_2' = distance from extreme compression fibre of the steel to the neutral axis.

a = distance of the centre of the compression steel reinforcing member from exterior compression surface of concrete.

o = distance of the centre of the tension steel reinforcing member from exterior tension surface of concrete.

rA_2 = area of normal section of reinforcing steel in tension.

$(1 - r)\,A_2$ = area of normal section of reinforcing steel in compression.

p_1 = intensity of compressive stress in the concrete at distance z from the neutral axis.

p' = intensity of tensile stress in the concrete at distance z from the neutral axis.

p_2 = intensity of stress in the steel at distance z from the neutral axis.

u = tensile or compressive strain in unit length of "fibre" at unit distance from the neutral axis.

In the theory of bending of concrete-steel beams it is assumed, as in the common theory of flexure, that any plane, normal section of the beam, before bending takes place, will remain plane (and normal) while the beam is subjected to bending. Hence:

$$p' = p_1 = E_1 u z, \text{ and } p_2 = E_2 u z. \quad . \quad . \quad . \quad 1.$$

The total stresses of tension and compression in any normal section of a beam induced by the bending must be equal to zero. The expression of this sum, written by the aid of Eqs. 1 and by which the neutral axis of the composite section is determined, is the following:

$$E_1 u \left\{ \int_0^{d_1} z\, d\, A_1 - \int_{h_1 - d_1}^{0} z\, d\, A_1 \right\} + E_2 u \int_{d_2' - h_2'}^{d_2'} z\, d\, A_2 = 0. \quad 2.$$

Or

$$\int_0^{d_1} z\, d\, A_1 - \int_{h_1 - d_1}^{0} z\, d\, A_1 + \frac{E_2}{E_1} \int_{d_2' - h_2'}^{d_2'} z\, d\, A_2 = 0. \quad . \quad 3.$$

Eq. 3 is perfectly general, and the position of the neutral

axis can always be located by it whatever may be the shape of cross-section of either the concrete or steel.

Fig. 1 may be taken as an arbitrary typical composite section showing the preceding system of notation applied to it. The outline of the concrete is rectangular, as in the ordinary concrete-steel beam. The steel in compression is represented as two steel angles, while three round rods constitute the steel in tension. In the next article the application of the general Eq. 3 to the special case of the ordinary concrete-steel beam will be made.

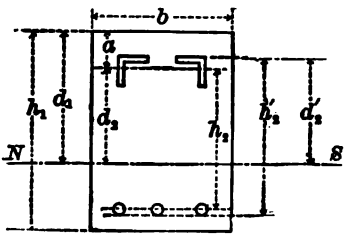

FIG. 1.

The general value of the bending moment of the stresses induced in any normal section of a composite beam can be at once written by the aid of Eqs. 1. The typical expression of the differential moment is:

$$p_1 \, d \, A_1 \, z = E_1 \, u \, z^2 \, d \, A_1.$$

Hence the general value of the moment is:

$$M = E_1 \, u \int_{h_1 - d_1}^{d_1} z^2 \, d \, A_1 + E_2 \, u \int_{h_2' - d_2'}^{d_2'} z \, d \, A_2. \quad . \quad . \quad 4.$$

Eqs. 3 and 4 cover the theory of bending or flexure of composite beams of two materials.

Art. 2.—Position of Neutral Axis in Ordinary Rectangular Concrete-steel Beams by Common Theory of Flexure.

The ordinary reinforced or concrete-steel beam is formed with a rectangular normal cross-section with a constant depth d_1 and a constant width b, the outline of the concrete section being rectangular and the steel reinforcement, or bars, being embedded in the concrete, all as typically shown in the figure on this page. The area of the steel section is small in comparison with that of the concrete, usually from about one to one

and one-half per cent. of the latter, so that there is no sensible error in making $d A_1 = b\, d\, z$. The first two terms of the first member of Eq. 3 of the preceding article then takes the form:

$$b\int_0^{d_1} z\, d\, z - b\int_{h_1 - d_1}^{0} z\, d\, z = \frac{b\, d_1^2}{2} - \frac{b\,(h_1 - d_1)^2}{2} = b d_1 h_1 - \frac{b\, h_1^2}{2} \quad . \quad . \quad 5.$$

As the steel section is always small it will be essentially correct to consider each portion of it concentrated at its centre of gravity, so that there may be written:

$$\int_{h_2' - d_2'}^{d_2'} z\, d\, A_2 = (1 - r)\, A_2\, d_2 - r\, A_2\, (h_2 - d_2) = A_2\, (d_2 - r\, h_2). \quad . \quad 6.$$

By introducing the values of Eqs. 1 and 2 in Eq. 3 of the preceding article the latter will become:

$$b\, h_1\, d_1 - \frac{b\, h_1^2}{2} + \frac{E_2}{E_1} A_2\, (d_2 - r\, h_2) = 0.$$

Or, since $d_2 = d_1 - a$,

$$\therefore d_1 = \frac{\frac{1}{2} h_1^2 + \frac{E_2}{E_1}\frac{A_2}{b}(a + r\, h_2)}{h_1 + \frac{E_2}{E_1}\frac{A_2}{b}} \quad . \quad . \quad . \quad . \quad 7.$$

As in the preceding case, when the steel reinforcement is wholly on the tension side of the beam it is necessary only to place $r = 1$ in Eq. 7.

If the entire steel reinforcement is on the tension side of the beam as shown in Fig. 2 and as often occurs, r will be unity in Eq. 3, or $1 - r = 0$. The only effect produced by this substitution in Eq. 3 is to make

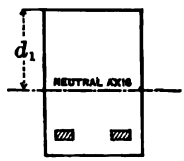

FIG. 2.

$a + r\, h_2 = a + h_2 =$ *distance from centre of steel bars to upper surface of beam.*

Neglect of the Tensile Resistance of Concrete.

Many civil engineers consider it advisable to neglect the bending resistance of that part of the concrete in tension and to

estimate the carrying capacity of the reinforced beam as if the compressive bending resistance constitutes the entire bending resistance of the concrete. In that case the stress condition assumed would be exhibited by Fig. 3. These stress conditions usually exist before rupture in consequence of the concrete in tension failing by cracking. Again, cracks may sometimes be formed, or at least be started, by a number of extraneous causes independent of the loading. For these reasons it will usually, or at any rate frequently, be the part of prudence and good practice to ignore the tensile bending resistance of the concrete in concrete-steel beams.

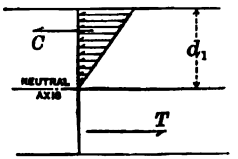

FIG. 3.

In order to find the neutral axis for this condition of stresses, it is only necessary to omit the second integral in the first member of Eq. 3 of the preceding article. This will give:

$$d_1^2 + 2\frac{A_2}{b}\frac{E_2}{E_1}d_1 = 2\frac{A_2}{b}\frac{E_2}{E_1}(a + r\,h_2).$$

By solving this quadratic equation:

$$d_1 = -\frac{E_2}{E_1}\frac{A_2}{b} \pm \sqrt{\left(\frac{E_2}{E_1}\frac{A_2}{b}\right)^2 + 2\frac{E_2}{E_1}\frac{A_2}{b}(a + r\,h_2)} \quad . \quad . \quad 8.$$

The value of d_1 in Eq. 8 gives the distance of the neutral axis below the upper or compressive surface of the beam when the tensile resistance of the concrete is neglected.

When the steel reinforcement is wholly on the tension side of the beam, $r = 1$, and $a + r\,h_2 = a + h_2$.

Art. 3.—Bending Moments in Concrete-steel Beams by Common Theory of Flexure.

By introducing the notation already employed into the ordinary formulæ of the common theory of flexure, the following general expression for the bending moment will result:

$$M = E_1 u \int_0^{d_1} z^2\,d\,A_1 + E_1 u \int_{h_1 - d_1}^{0} z^2\,d\,A_1 + E_2 u \int_{h_2' - d_2'}^{d_2'} z^2\,d\,A_2. \quad 9.$$

The first two integral expressions in the second member of Eq. 9 are moments of inertia of the two portions of A_1, about the neutral axis of the section, while the third is the moment of inertia of A_2 about the same axis. That equation may then be written:

$$M = E_1 u I' + E_1 u I'' + E_2 u I_2. \quad . \quad . \quad . \quad . \quad 10.$$

If I_1 is the moment of inertia of A_1 about the neutral axis, then $I' + I'' = I_1$.

Again, if k_1 is the greatest intensity of compression in the concrete and k' the greatest intensity of tension, while c is the intensity of compressive stress in the steel at the distance d'_2 from the neutral axis, then:

$$E_1 u = \frac{k_1}{d_1}, \quad E_1 u = \frac{k'}{h_1 - d_1}, \quad \text{and } E_2 u = \frac{c}{d_2}. \quad . \quad . \quad 11.$$

By introducing these values in Eq. 10:

$$M = \frac{k_1}{d_1} I' + \frac{k'}{h_1 - d_1} I'' + \frac{c}{d_2} I_2. \quad . \quad . \quad . \quad 12.$$

This is the general value of the bending moment for any form of section or any two materials. The rectangular normal section will be employed for the outline of the concrete and steel bars of any desirable shape, but of comparatively small section, for concrete-steel beams. The corresponding quantities are then to be introduced into Eq. 12. For this case:

$$I' = \frac{b\, d_1^3}{3} \text{ and } I'' = \frac{b\,(h_1 - d_1)^3}{3}.$$

Also:

$$I_2 = (1 - r)\, A_2\, d_2^2 + r\, A_2\, (h_2 - d_2)^2.$$

The value of I_2 may be written in this form for the reason that the steel sections (A_2) are small.

The insertion of these quantities into Eq. 12 will give:

$$M = k_1 \frac{b\, d_1^2}{3} + k' \frac{b(h_1 - d_1)^3}{3} + c \left[(1 - r) A_2 d_2 + r A_2 d_2 \left(\frac{h_2}{d_2} - 1 \right)^2 \right]. \quad 13.$$

The use of this equation for the value of the bending moment

can be made more convenient if its second member be changed in form so as to involve but one intensity of stress, and that one may preferably be the greatest compression, k_1, in the concrete.

From Eq. 11:

$$k' = \frac{h_1 - d_1}{d_1} k_1, \text{ and } c = \frac{E_2}{E_1}\frac{d_2}{d_1} k_1. \quad . \quad . \quad . \quad 14.$$

By introducing these values in Eq. 13:

$$M = k_1\left(\frac{bd_1^2}{3} + \frac{b}{d_1}\frac{(h_1-d_1)^3}{3} + \frac{E_2}{E_1}\frac{d_2^2}{d_1}A_2\left[(1-r)+r\left(\frac{h_2}{d_2}-1\right)^2\right]\right). \quad 15.$$

This value of the bending moment M requires only the greatest intensity of compression, k_1, in the concrete to be given, all other quantities in the second member being either given or assignable.

The greatest intensity of tension in the steel will be:

$$t = \frac{h_2 - d_2}{d_2} c = \frac{E_2}{E_1}\frac{h_2 - d_2}{d_1} k_1 \quad . \quad . \quad . \quad . \quad . \quad 16.$$

Thus all the intensities in the extreme fibres may be expressed in terms of any one. As a practical matter it is probably preferable to use the greatest compression in the concrete for that purpose, as will appear later.

If the steel reinforcement is wholly on the tension side of the beam, $r = 1$, and Eq. 15 will take the form:

$$M = k_1\left(\frac{bd_1^2}{3} + \frac{b}{d_1}\frac{(h_1 - d_1)^3}{3} + \frac{E_2}{E_1}\frac{A_2}{d_1}(h_2 - d_2)^2\right) \quad . \quad 17.$$

Neglect of the Tensile Resistance of Concrete.

If the bending resistance of the concrete on the tension side of the neutral axis be neglected it is necessary only to omit the second term of the second member of Eq. 15. That equation will then take the form:

$$M = k_1\left(\frac{bd_1^2}{3} + \frac{E_2}{E^1}\frac{d_2^2}{d_1}A_2\left[(1-r)+r\left(\frac{h_2}{d_2}1 - \right)^2\right]\right). \quad 18.$$

In case the steel reinforcement is wholly on the tension side of the beam, $r = 1$, and the bending moment becomes:

$$M = k_1 \left(\frac{bd_1^3}{3} + \frac{E_2}{E_1} \frac{A_2}{d_1} (h_2 - d_2)^2 \right) . \quad . \quad . \quad . \quad 19.$$

In all the preceding formulæ the distance d_1 of the neutral axis from the exterior compression surface of the beam is to be found by the appropriate equation in the preceding article, d_1 from Eq. 7 or Eq. 8 of that article being required for use in the general Eq. 15 of this.

The preceding equations complete all that is necessary in the treatment of practical questions of design or of ultimate carrying capacity.

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